Comments on the exam will be collected here. This file may grow if there are more questions about particular problems.

First Part

Problem 1 The derivative of a polynomial should be so **automatic** that you can do it **repeatedly** without difficulty. Such functions allow your interpretation of the notation for second derivatives to be tested without introducing difficulties of preparing your expression for the first derivative to be differentiated.

Problem 2 Several rules are combined here. The **chain rule** gives a product of the derivatives of the two functions **composed** here. Since the expression is a power of **something**, the first factor is the familiar "exponent times one lower power" of **the something**. The second factor is the derivative of the something. (This much is sometimes considered to be an extended power rule. It is useful if you are aiming to develop the skill to write derivatives without leaving any clues about your analysis of the given function. Since your exam papers will be graded by someone who does **not** assume that you know Calculus, it would be better to give **more** details.)

In this problem, the second factor is the derivative of a quotient (of linear expressions). The quotient rule should be used. This derivative simplifies to a constant divide by the square of the original denominator. Since this simplification is **easily done** and gives a **clear improvement**, it should probably be done, although it is not necessary.

Problem 3 This is a example of **implicit differentiation**. You are given an **equation** defining y as a function of x, rather than an expression to be fed to the differentiation rules. As long as you believe that everything depends on x and the given equation is an **identity**, the chain rule will lead to a correct new equation relating dy/dx to x and y. You can then solve for dy/dx. (This will always be easy because the equation is **linear** in dy/dx.)

Second Part

Problem 1

Grades on this were high, but that does not mean that a similar problem on the final exam will also have high grades. The main things I was looking for were: (1) a correct expansion of $f(x+h)$; (2) a simplification of the **difference quotient**; (3) indication that the definition involves a limit.

Grading may be stricter on the final exam. This type of exercise requires you to show understanding of the definition of the derivative. Since you need to **convince the grader** that **you understand the subject**, which is more than is required from anyone who has already been certified as knowing the subject, you should aim for a **complete** demonstration without being verbose. This is much easier to do if you write a sequence of **complete mathematical sentences**, such as equations that assert that two quantities really are equal. It is difficult to do this in a convincing way if your work follows the pattern used to solve equations in elementary algebra, since this often degenerates into a sequence of unnamed algebraic transformations, with quantities connected by equal signs even if they aren't equal.

It should also be noted that the difference quotient is only defined if $h \neq 0$.

Problem 2 This is a **related rates** problem, which **requires Calculus**. In this problem there are **three** distances: the two distances along the roads and the distance **between** the vehicles. These are related by the **Pythagorean theorem**. The **rates** are the derivatives of these distances **with respect to time**. There is no assumption that any rates are constant, but only **instantaneous** rates appear when the identity given by the Pythagorean theorem is differentiated. This reinforces the observation made several times so far in the course that derivatives (or differentials) are to be considered as **new variables** that should not be combined with other quantities used in the same scenario.

In order to make your work readable, it is useful to begin by following the style of modern programming languages by **declaring your variables**. That is, each symbol that you use should represent a number that is identified as a particular measurement with respect to **clearly identified units**. All terms in an equation relating these quantities should represent **consistent measurements**. The three distances in this problem are related by the Pythagorean Theorem in which the terms are all measurements in **square feet**.

Differentiating this with respect to time gives another equation relating measurements in **square feet per second**.

Problem 3 You were not given an analytic expression for the function, so the answers were to be based on a **reasonable interpretation** of the graph. It looks like it could be drawn without removing the writing instrument from the paper. This is an informal way to say that there are **no jumps**. There are other types of discontinuities, but they are even more visually disturbing. This function is **continuous on the whole domain**.

A jump in the graph of $f'(x)$ leads to a **corner** in the graph of $f(x)$. The given graph has corners at $(0, 0)$ and $(1, 1)$. Thus, there is no derivative at $x = 0$ or $x = 1$. A derivative exists at other values of x.

To estimate the derivative, draw a tangent line and use the given grid to determine the slope of the line you drew.

Since the graph is a straight line of slope -1 between -1 and 0, the derivative is -1 there, and since the function is constant for $x > 1$, its derivative is zero for such x. You can determine the reliability of your graphical technique for x between 0 and 1 by comparing your estimate with the value of $f'(x)$ that you can find after being told that $f(x) = x^2$ for $0 < x < 1$.

Problem 4 These two problems explore one aspect of limits. In both parts, the denominator is x minus the point at which the limit is to be evaluated.

In one part, this denominator divides the numerator so the quotient is a polynomial. Except for this special value, the function is given by a simpler expression. The language of limits calls the value of the simpler expression at the special value the **limit** of the original expression as one **approaches** the special value.

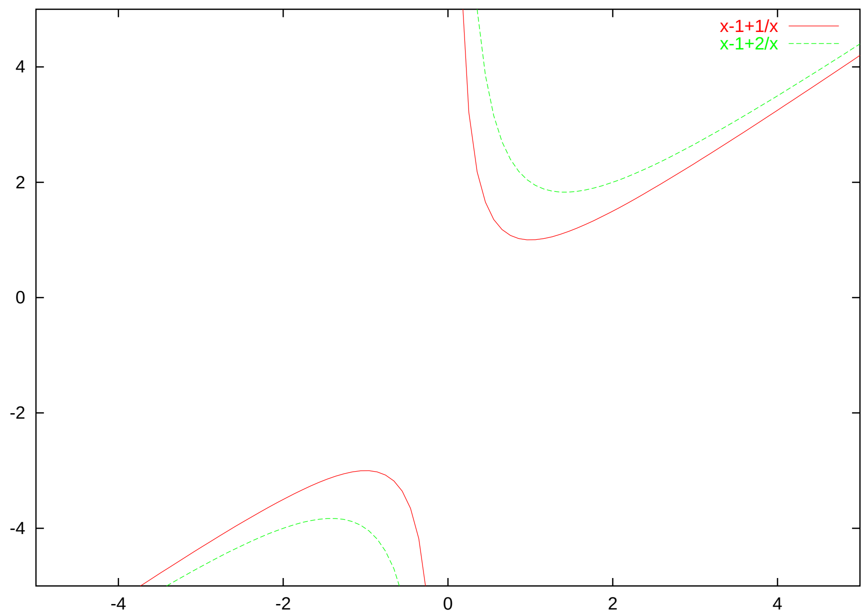
In the other part, attempting to divide the numerator by the denominator leaves a remainder. Thus, except for the well-behaved quotient, the given fraction behaves like a constant divided by the difference between x and the special value. If x is close to the special value, the denominator is small while the numerator is bounded away from zero. **Such quotients are large.** This limit does not exist because values near the special value will be far from any number.

Some answers were based on trying to factor the numerator rather than divide by the denominator. This **works**, but **only** because there is unique factorization of polynomials. It **didn't take forever** because everything was of very low degree. In general, **factoring is difficult** and should be avoided. In this problem, it only gives indirect evidence for something that is easily studied **directly**.

Your calculator can help you with questions like this. If you graph the function near the special value, you can see whether the function **seems to have** a value there, which is a sign that a limit exists. If you **program** the calculation of the function, you can easily evaluate it near the special value. You might even try evaluating it **at** that value. (I have an HP48G and want to concentrate on learning its approach to advanced mathematics, so I have not tested the TI-8x family of calculators. For these exercises, both expressions give errors at the special value, but the errors are **different**. The numerator and denominator of the fraction are also shown.)

When direct evaluation of the numerator and denominator of a fraction **appears to give** $0/0$, one expects to find a **simple** common factor in both parts. Removing this factor from both parts leaves a fraction that is equal to the original wherever the common factor isn't zero.

Problem 5 The mystery of the misalignment of the graph in this problem has been solved. The difficulty was noticed between the printing of the exam and its use, so I could warn you that a **small** inaccuracy could be expected. During the exam, a student noticed that the discrepancy was significant when $x_0 = 1$. Armed with this information, it was possible to determine that the graph of a different function (that had been considered for use on the exam, but rejected) was included. It was harder to track this down because the **key** giving the formulas used in the plot was disabled (graphs produced by this program have been used in postings based on lectures, and those include a key) and no log was kept of the instructions leading to the graph. The function actually graphed on the exam was $x - 1 + 2/x$ instead of $x - 1 + 1/x$. Here is a graph showing **both** graphs, and including a key to identify the formulas used to draw them.



Notice that the curves have the same general appearance and are close except near the points where there are horizontal tangents, so there would be little difficulty at other points. The main difficulty in these problems arises from using the **general expression** for dy/dx instead of its value at x_0 in the role of the slope. The resulting expression will **not** be the equation of a line. By asking for a sketch of the line, we hoped to encourage you to interpret your answer to part (a) and assure that it described a line.

Problem 6. This model was **quoted** from one of the early exercises in the textbook, using the same numbers and referring to the same product (digital TV), with the same scale for the variables. The intent was to survey all topics touched on in the course. There was no calculus in the problem.

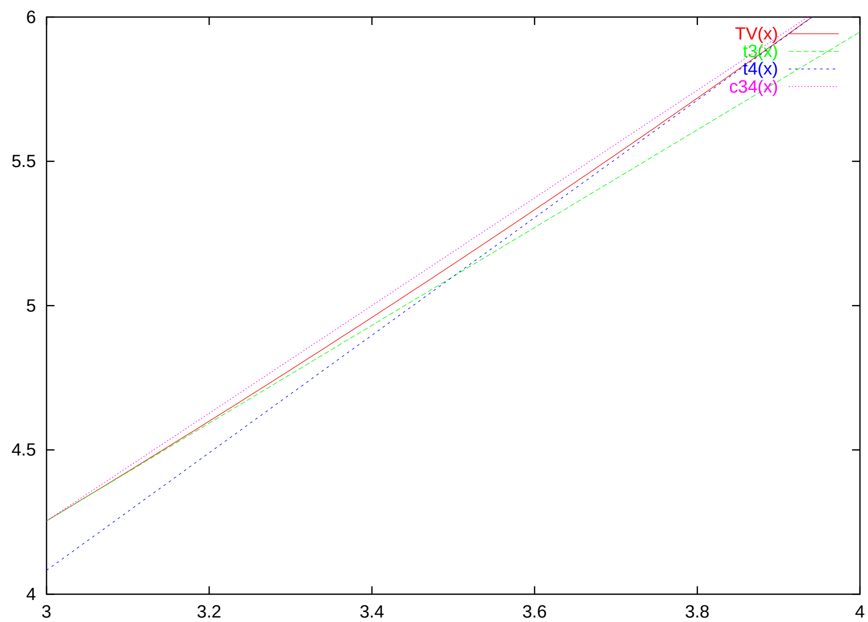
Several features of models are illustrated here.

In order to be able to use **small numbers** in the computation, $t = 0$ was made to describe the start of time interval being studied instead of reading a number off the calendar. The use of small numbers has more than simply cosmetic value: evaluation of polynomials becomes inaccurate if the terms of the polynomial can be large while the total is not.

The time interval also has an upper limit. Simple formulas constructed to match observations over a small interval do not often **extrapolate** much beyond the smallest interval containing those observations. Here, the model appears to have been constructed to **predict** a market. Once it was determined that a cubic polynomial would give a reasonable prediction, the coefficients were found by measuring some **consequences** of the prediction that could be measured in the initial months of the interval.

Calculus does play a role in studying these consequences, since a formula for one measurable quantity automatically leads to formulas for the derivative (and second derivative, etc.) of that quantity. Information about the derivative can then be used to help describe the original function.

In this case, the formula is simple enough to be evaluated at any value of t, and that was all that was intended. There are also ways to **approximate** the value at one point using information at nearby points. For example, the idea of **differentials** is to use the derivative to relate **small** changes in two quantities. This is equivalent to following a tangent line at $t = t_0$, using the value on the line when t is a nearby value t_1 to approximate $f(t_1)$. Alternatively, if some kind soul has given you the values of $f(t)$ for all integer values of x in the domain of f, you can use **linear interpolation** to approximate $f(t)$ at other values of t. This is equivalent to locating the point on the line segment connecting given values, which is a chord of the given graph. For this particular model, the tangent lines will all be below the graph and the chords will be above the curve. (We will explore this later as an application of the second derivative. Here is a graph showing the given function between $t = 3$ and $t = 4$ with the tangent lines at both ends of this interval and the chord joining those points. Other intervals would be similar.



Although these approximations were used by some students, they give no advantage in such a simple model because the given f is so easy to evaluate — perhaps even easier than the approximations.)

Problem 7. The formula sheet contained the statement that $R = xp$. To get a formula in terms of a **single independent variable**, you can use either the given expression for x in terms of p to get R in terms of p, or you can solve the given equation to get p in terms of x and **multiply this** by x to get R in terms of x. (Failure to multiply by x was a common error in spite of having the formula for R available.)

Since the formula sheet mentioned making R a function of x alone, more solutions used this approach although the expression for R in terms of p is simpler. Thus, for

$$x = \frac{1}{5}(225 - p^2),$$

the expressions for R are

$$R = \frac{p}{5}(225 - p^2) = \frac{1}{5}(225p - p^3)$$
$$R = x\sqrt{225 - 5x} = \sqrt{225x^2 - 5x^3}$$

The word "**marginal**" in economics means **derivative**, but we have two different choices of independent variable. In order to have a **precise** definition, we need to know which one is the **intended** independent variable. To decide this, we need to recall the initial discussion of this term. The idea was that often economic analysis is concerned with the effect on such quantities as **cost**, **revenue**, and **profit** of **making one more item**. The difference in the dependent variable when x changes by 1 is a difference quotient that is approximated by the derivative with respect to x. In particular, the custom of writing quantities as functions of x aims at obtaining marginal quantities as explicit derivatives.

Once you know that you are looking for dR/dx, the technique of implicit differentiation allows it to be found as $(dR/dp) \big/ (dx/dp)$. Thus

$$\frac{dR}{dx} = \frac{\frac{1}{5}(225 - 3p^2)}{\frac{1}{5}(-2p)} = \frac{3p^2 - 225}{2p}.$$

If you express R in terms of x, you can differentiate directly. I used the intermediate expression and the product rule, but one student used the final expression given above, which leads immediately to a simplified form of the derivative. Thus,

$$\frac{dR}{dx} = \frac{1}{2}(225x^2 - 5x^3)^{-1/2}(450x - 15x^2) = \frac{450x - 15x^2}{2x\sqrt{225 - 5x}} = \frac{450 - 15x}{2p}.$$

You should be able to check that this agrees with the previous answer.

None of the work or results of the first two parts is used in the rest of the problem, although correct results in one part can be used to check correct results in another. Statements on the formula sheet may be interpreted as saying the **elastic demand** corresponds to $dR/dp < 0$. Since $dp/dx < 0$, this is equivalent to $dR/dx > 0$.

A formula for E is given, so it should be **transcribed accurately** as the first step of the solution. In far too many cases, the sign was lost. This sign has a peculiar role: although it is visibly a minus sign, its role is to negate the negative quantity $f'(p)$ to get a positive value for E. In particular, a negative value for E only shows that a mistake was made. It cannot be used to conclude that demand is inelastic. Since $f(p)$ is identified in the description of the formula for E as the function giving x in terms of p, we have $f(p) = (225 - p^2)/5$ and $f'(p) = -2p/5$. Thus $E = 2p^2/(225 - p^2)$. Since values of p are given in (d), this is the most useful expression for E. Note that it is nonnegative since we have required that $0 \leq p \leq 15$. Solving the inequality $E > 1$ gives elastic demand if and only if $2p^2 > 225 - p^2$, if and only if $3p^2 > 225$, if and only if $p^2 > 45$, if and only if $p > \sqrt{45} \approx 6.7$.