Exercises from Section 2.1

Exercises 2, 3 and 10 interpreting function notation were shown, as well as exercise 24 in which the domain was to be found. Here we give Statement Work Answer.

Exercise 2. Given

$$f(x) = 4x - 3,$$

find the following:

$$f(4) = 4(4) - 3 = 16 - 3 = 13$$
$$f(1/4) = 4(1/4) - 3 = 1 - 3 = -2$$
$$f(0) = 4(0) - 3 = 0 - 3 = -3$$
$$f(a) = 4(a) - 3 = 4a - 3$$
$$f(a+1) = 4(a+1) - 3 = 4a + 4 - 3 = 4a + 1$$

Although we claim to allow **anything** to be substituted for the variable, things like $f(a+1)$ might be better explained as examples of composition of functions (defined in Section 2.2).

Exercise 3. Given

$$g(x) = 3x^2 - 6x - 3,$$

find the following:

$$g(0) = 3(0)^2 - 6(0) - 3 = 0 - 0 - 3 = -3$$
$$g(-1) = 3(-1)^2 - 6(-1) - 3 = 3 + 6 - 3 = 6$$
$$g(a) = 3(a)^2 - 6(a) - 3 = 3a^2 - 6a - 3$$
$$g(-a) = 3(-a)^2 - 6(-a) - 3 = 3a^2 + 6a - 3$$
$$g(x+1) = 3(x+1)^2 - 6(x+1) - 3 = 3x^2 + 6x + 3 - 6x - 6 - 3 = 3x^2 - 6$$

Although it is common to use a definition of $g(x)$ to find $g(x+1)$, the use of the same letter x in two different ways may be confusing. In particular, to consider this usage as an example of composition may be an *abuse of notation*.

Exercise 10. Given

$$g(x) = \begin{cases} -\frac{1}{2}x + 1 & \text{if } x < 2 \\ \sqrt{x-2} & \text{if } x \geq 2 \end{cases}$$

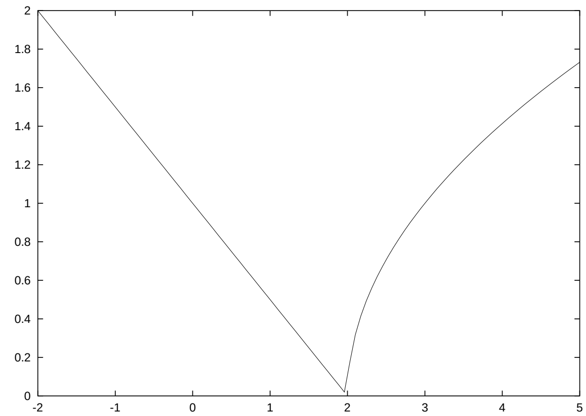
find the following:

$$g(-2) = -\frac{1}{2}(-2) + 1 = 2$$
$$g(0) = -\frac{1}{2}(0) + 1 = 1$$
$$g(2) = \sqrt{2-2} = 0$$
$$g(4) = \sqrt{4-2} = \sqrt{2}$$

where the first two lines use the first case of the definition of $g(x)$ and the last two lines use the second case of the definition. The inequalities $x < 2$ and $x \geq 2$ appearing in this **definition by cases** are such that **exactly one** holds for any real number x. The calculation of the function begins by determining which case applies. One this is done, there is a single expression that is to be used to determine the value of the function. Although this type of definition has been common in advanced mathematics for some time, it is not often seen in school mathematics, and may not be easy to describe on your calculator.

In this case, each expression for $g(x)$ is for all **relevant** x, so the domain of g is **all real numbers**.

In class, someone asked how the range of g could be found. The method for doing this was to first find the graph of $g(x)$:



This picture indicates that $g(x) \geq 0$ for all x; and if $y \geq$, then there is an x with $g(x) = y$. The first part uses algebraic properties of inequalities to show the $x < 2$ implies $-{}^{1}\!/_{2}x + 1 > 0$ (for $x \geq 2$, it is only necessary to note that the $\sqrt{x-2}$ is defined to be the **positive** solution y of $y^2 = x - 2$. To show that the range is all nonnegative real numbers, try to solve $y = g(x)$ for x. The definition by cases demands some care, but it is not difficult to show that $g(x) = 0$ only for $x = 2$ and that each positive value is $g(x)$ for exactly one $x < 2$ and exactly one $x > 2$.

Exercise 24. What is the domain of

$$f(x) = \sqrt{x-5}?$$

This is a case where an **implicit domain** must be found. That is, we attempt to evaluate the given formula for all real numbers x and note any **obstruction** to the calculation succeeding. There is no difficulty getting $x - 5$ from x, since subtraction is always defined, but the next step takes a square root, and **square roots can only be taken of nonnegative quantities**. Thus the evaluation of this expression requires

$$x - 5 \geq 0,$$

which can be solved to give

$$x \geq 5$$

and the set where this is true can be expressed as the interval $[5, \infty)$.

Note: This is the usual type of Find the domain exercise. The idea is to break the evaluation of the function into small pieces and to **flag** any step that may cause the calculator to signal an error. Taking the square root (or **logarithm**) of a negative number is one example. **Dividing by zero** is another example. Otherwise, the standard operations are defined.

The discussion on page 62 of the textbook, didn't say anything about this type of exercise. Instead, it featured Example 3, in which there were additional **feasibility** requirements for an expression to describe a geometric construction. Although the formula for the volume can always be evaluated, it only corresponds to this physical construction on a particular interval. This will be important later when we find the largest box that can be found by this construction from a given piece of cardboard.

Exercises from Section 2.2

Exercise 35. Given

$$h(x) = (2x^3 + x^2 + 1)^5,$$

we are to write $h = g \circ f$. This **analysis** is usually done by working with **expressions** rather than **functions**. That is, introduce the name z for $h(x)$ and look for some way in which the expression can be computed by first finding some expression y that depends only on x that is repeated in z and such that z can be expressed in terms of y with no additional contribution from x. Then f and g are described by $y = f(x)$ and $z = g(y)$. There is no mechanical procedure to do this, but these questions are only asked by someone who has just defined a function h as a composition. Thus, the question really is, "What do you think I did to create this function?" The use of different variables in the description of f and g emphasizes the analysis of the expression for $h(x)$.

In this case, one sees $z = (2x^3 + x^2 + 1)^5$, so if $y = 2x^3 + x^2 + 1$, then $z = y^5$. This leads to

$$\begin{aligned} f(x) &= 2x^3 + x^2 + 1; \\ g(y) &= y^5. \end{aligned}$$

Exercise 42. Here,

$$z = \frac{1}{\sqrt{2x+1}} + \sqrt{2x+1}$$

The computation is seen to involve the following steps

$$\begin{aligned} u &= 2x, \\ v &= u + 1, \\ y &= \sqrt{v}, \\ z &= \frac{1}{y} + y. \end{aligned}$$

Since the exercise asks to express z as a compositon of **two** functions, you should choose one of the intermediate variables u, v or y and let f be the expression of your chosen variable in terms of x and g be the expression for z in terms of your chosen variable.