

Exercises from Section 2.3

Exercise 32. This problem is assigned for homework, so I will confine my remarks to aspects of the topic not developed in the exercise that illustrate features of mathematical models.

The function in this exercise is given as

$$f(t) = \begin{cases} 8.37t + 7.44 & \text{if } 0 \leq t \leq 8 \\ 2.84t + 51.68 & \text{if } 8 < t \leq 30 \end{cases}.$$

This function associates to a number t a value $f(t)$ which is also a number. The interpretation is based on relating these numbers to particular **measurements**. In particular, t is time in **years since 1970**. That is, it associates time with a **number line** by giving an **origin** — the year 1970 — and a **unit** — the year. Note that the origin is at the start of period of observation, not the beginning of some calendar thousands of years earlier. Other choices of unit would change the horizontal scale of the graph; other choices of origin would translate the graph of $y = f(x)$ horizontally with respect to the coordinate axes.

Similarly, $f(t)$ is another measurement, a **price** in dollars per kilo. Changing this unit would change the vertical scale on the graph.

When drawing this curve, the point $(0, 0)$ should be taken as the lower left corner of the graphing window. The right side should be $t = 30$, and the top should be a convenient value at least as large as $f(30) = 136.88$. Although the numerical value of the second coordinate at the upper right corner is more than four times the numerical value of the first coordinate, independent scaling allows this to fill a graphing window in which the true height is smaller than the true width (as it is on most calculators).

Although $f(8)$ is to be computed using the first line of the definition, the second line gives the same value. The significance of this will appear in Section 2.5.

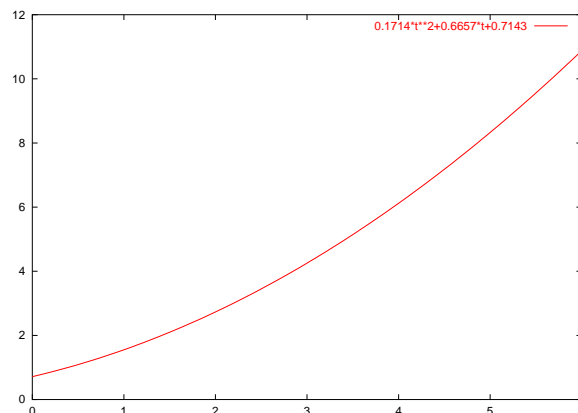
The graph shows linear growth over many years, but an abrupt change in the rate of that growth in the year 1978 ($t = 8$). This suggests that there was a change in the nature of the market at that time caused by a political event.

A graph is not included here since you are asked to draw one in one part of the exercise.

Exercise 33. The market for digital TV in the years 2000 through 2006 is predicted to grow according to the formula

$$f(t) = 0.1714t^2 + 0.6657t + 0.7143 \quad (0 \leq t \leq 6).$$

Here is a graph of this function:



The increased growth caused by the second degree term reflects an assumption that the superior features of this product will generate increased demand as people observe it in the homes of friends and neighbors. Since this model deals entirely with **predictions**, this assumption is likely to be based on observations of **related products**.

Again, t represents time in years from some starting year. This time $f(t)$ is measured in **millions of homes**. Although it claims to **count** something, this scale is such that a variation of a **few thousand** would not be seen on the graph (graphical accuracy is rarely better than 1/1000 of the range shown). This allows a smooth curve to represent something that (on some scale) only takes integer values.

Presumably, the four decimal places of each of the coefficients was the result of some **very expensive market research**.

Exercises from Section 2.4

Having called attention to **limits**, the next step is to give some methods for **finding** them. Exercises in the range 23–40 represent the easiest examples. In these, $f(a)$ can be calculated from the given expression for the function f and one has no reason not to expect

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Functions with this property at every a in their domain are called **continuous**. Although continuity is the key to finding some limits, it is customary in Calculus textbooks to only introduce it after discussing limits.

There is no way to illustrate this type of problem so that limits **appear to be used** in finding the answer, so no examples will be given here.

The example of $f(x) = x/x$ with $a = 0$ used in the prepared slides illustrates the next level of difficulty. Here $f(a)$ cannot be computed, but there is a function that is equal to f **everywhere else**, and it is the value of **that function** at $x = a$ that turns out to be the limit. We give two examples of fractions in which there is a common factor in numerator and denominator whose quotient can be treated as if it were 1, and then give another example in which additional algebra is needed to reduce the given function to this form. Finally, an example of **horizontal asymptotes**, also called **limits at infinity**, will be given.

Exercise 49.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2$$

If $x \neq 1$, it is possible to divide by $x - 1$, so the given fraction may be replaced by the quotient. Evaluating the quotient at $x = 1$ gives the value that **can be proved** to be the limit. (The second part of this solution is also an example of the type of exercise that was skipped).

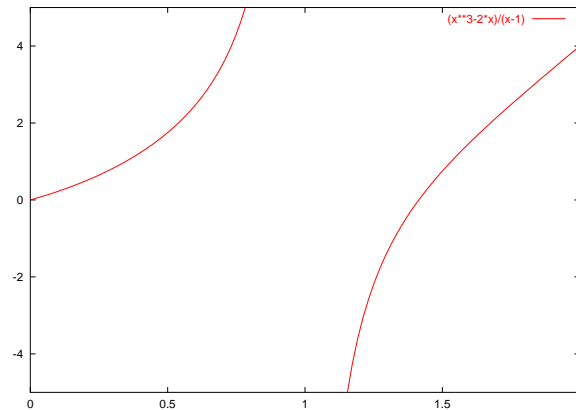
Extra exercise done in lecture.

$$\lim_{x \rightarrow 1} \frac{x^3 - 2x + 1}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x - 1) = 1$$

In lecture, the **long division** to justify the first step was shown, but that isn't easy to produce in type.

It was also noted that, if the function were $(x^3 - 2x)/(x - 1)$ where long division leads to both a quotient and a **remainder**, there would be no limit. The graph reveals a **vertical asymptote** at $x = 1$, so the

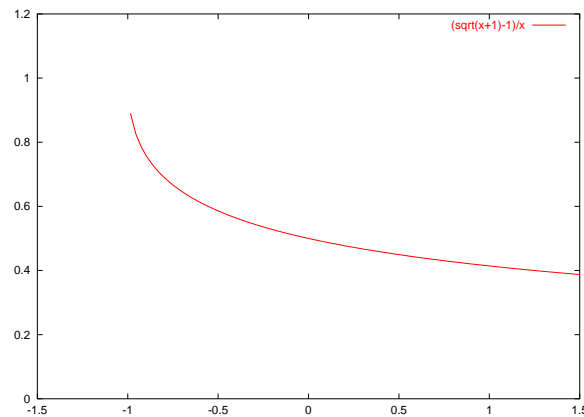
function is not close to any finite value there.



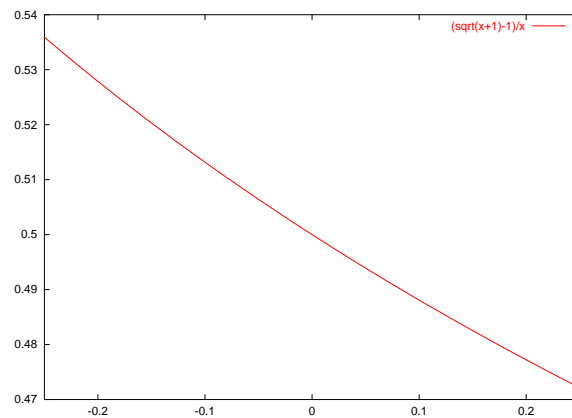
Exercise 7, p. 132 (Technology). Use a graph to estimate the limit.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

The domain of this function is all $x \geq -1$ (because of the square root) **except** $x = 0$ (because of the denominator). Here is a graph of this function:



In this view, the graph comes to an abrupt halt at the point $(-1, 1)$. Zooming in on a smaller interval around $x = 0$ gives



The magnification has made this portion of the graph look **much straighter** than the first view. This observation is at the heart of the **Differential Calculus** that we have been leading up to. The pictures suggest a limit of $1/2$ at $x = 0$, but my calculator gave 0.492. Calculators have no special knowledge about what they claim to be evaluating, and expressions that need to be evaluated as limits are often troublesome when evaluated at nearby points on a calculator. This is caused by a loss of accuracy evaluating part of an expression (in this case, the numerator), which can introduce unexpected errors.

The algebra that allows this expression to be written in a form that reveals the limit is

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \\ &= \lim_{x \rightarrow 0} \frac{(x+1) - 1}{x(\sqrt{x+1} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1} \\ &= \frac{1}{1+1} = \frac{1}{2}\end{aligned}$$

Example 6 on p. 120 found a similar limit.

Exercise 75.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^3 + x^2 + 1}{x^3 + 1} &= \lim_{x \rightarrow \infty} \frac{3x^3 + x^2 + 1}{x^3 + 1} \cdot \frac{x^{-3}}{x^{-3}} \\ &= \lim_{x \rightarrow \infty} \frac{3 + x^{-1} + x^{-3}}{1 + x^{-3}} \\ &= \frac{3}{1} = 3\end{aligned}$$

Here, numerator and denominator are multiplied by the multiplicative inverse of the highest power of x in the denominator. In the resulting expression, the denominator is bounded as $x \rightarrow \infty$. The behavior of the fraction is determined by the behavior of the numerator of this new fraction.

Since negative powers of x go to zero when x is large, the idea of this method is to rewrite the expression in an equivalent form that uses negative powers apart from some constant terms.

Exercise 79.

$$\lim_{x \rightarrow \infty} \frac{x^5 - x^3 + x - 1}{x^6 + 2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{x^{-1} - x^{-3} + x^{-5} - x^{-6}}{1 + 2x^{-2} + x^{-6}} = 0.$$

Here, numerator and denominator were multiplied by x^{-6} since the highest power in the denominator is x^6 .

For rational functions, we can refer to the limit as $x \rightarrow \infty$ because such limits are found by characterizing this as $x^{-1} \rightarrow 0$. Other functions can have different behavior as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. (This was inserted in response to a question from a member of the class).

The extra ink required to write $+\infty$ to signify **large and positive** while using the unadorned ∞ for **of large absolute value** seems to pay large dividends in clarity. It is curious that this is not used in the textbook although **one-sided limits** at finite values do have a special notation (introduced in Section 2.5).