Exercises from Section 3.3 Although many of these exercises can be done by expanding a pattern written on the basis of the **general power rule**, the method employed here will be the introduction of one variable to represent the given expression and a new variable to represent an intermediate expression followed by use of the **chain rule** in the form

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

You are encouraged to learn, at your own pace, methods for quickly finding derivatives, but speed should not be allowed to replace accuracy or interfere with understanding. Calculators have already been programmed to find derivatives, provided that the function is described correctly. As the use of calculators for this purpose gains acceptance, fast human calculation will be less important than the ability to notice when a wrong answer indicates that the question was incorrectly presented to the calculator.

Exercise 1. $z = (2x-1)^4$. Thus, $z = y^4$ with $y = 2x - 1$. Then

$$\frac{dz}{dy} = 4y^3 = 4(2x-1)^3 \text{ and } \frac{dy}{dx} = 2.$$

The chain rule gives

$$\frac{dz}{dx} = 8(2x-1)^3.$$

As a check, the binomial theorem shows that

$$z = 16x^4 - 32x^3 + 24x^2 - 8x + 1,$$

so that term-by-term differentiation gives

$$\frac{dz}{dx} = 64x^3 - 96x^2 + 48x - 8 = 8(8x^3 - 12x^2 + 6x - 1),$$

which is the expansion of the result obtained using the chain rule.

Exercise 3. $z = (x^2+2)^5$. Thus, $z = y^5$ with $y = x^2 + 2$. Then

$$\frac{dz}{dy} = 5y^4 = 5(x^2+2)^4 \text{ and } \frac{dy}{dx} = 2x.$$

The chain rule gives

$$\frac{dz}{dx} = 10x(x^2+2)^4.$$

As a check, the binomial theorem shows that

$$z = x^{10} + 10x^8 + 40x^6 + 80x^4 + 80x^2 + 32,$$

so that term-by-term differentiation gives

$$\frac{dz}{dx} = 10x^9 + 80x^7 + 240x^5 + 320x^3 + 160x = 10x(x^8 + 8x^6 + 24x^4 + 32x^2 + 16),$$

which is the expansion of the result obtained using the chain rule. Notice that z expands to a polynomial with all terms of **even** degree, so dz/dx will have all terms of **odd** degree. This requires a factor of x with complementary factor again containing only terms of even degree. This is, apart from constant factors, the factorization given by the chain rule.

Exercise 14. $z = \sqrt{2x^2 - 2x + 3}$. Thus, $z = y^{1/2}$ with $y = 2x^2 - 2x + 3$. Then

$$\frac{dz}{dy} = \frac{1}{2}y^{-1/2} = \frac{1}{2y^{1/2}} = \frac{1}{2\sqrt{2x^2 - 2x + 3}} \text{ and } \frac{dy}{dx} = 4x - 2.$$

The chain rule gives

$$\frac{dz}{dx} = \frac{4x - 2}{2\sqrt{2x^2 - 2x + 3}} = \frac{2x - 1}{\sqrt{2x^2 - 2x + 3}}.$$

Normally, you would **not** do the last step on an exam unless you were told to simplify the answer. If only the calculus is being tested, you should avoid extra work that would not improve your score if done correctly but for which you would be penalized in the event of an error. It is included here because at the end of the course we will consider the problem of recovering z from dz/dx and, if dz/dx were given in simplified form, the factors of 2 in numerator and denominator that appear here would need to be discovered somehow.

Exercise 22. $z = (5t^3 + 2t^2 - t + 4)^{-3}$. Thus, $z = y^{-3}$ with $y = 5t^3 + 2t^2 - t + 4$. Then,

$$\frac{dz}{dy} = -3y^{-4} = -3(5t^3 + 2t^2 - t + 4)^{-4} \text{ and } \frac{dy}{dt} = 15t^2 + 4t - 1.$$

The chain rule gives

$$\frac{dz}{dt} = -3(15t^2 + 4t - 1)(5t^3 + 2t^2 - t + 4)^{-4}.$$

Notice that the exponent of y is $-3 - 1 = -4$. The derivatives of functions containing negative exponents have exponents that are **slightly more negative**.

Unless you are given a good reason to do so, the derivative should not be written as

$$\frac{-45t^2 - 12t + 3}{(5t^3 + 2t^2 - t + 4)^4},$$

since the problem was given in terms of negative exponents and not as a fraction.

Exercise 24. $z = (2t - 1)^4 + (2t + 1)4$. This problem contains a new feature. The expression for z in terms of t is a **sum** to which we can apply the **sum rule**, and only when that requires us to look at the individual terms do we see expressions to which the generalized power rule applies. Thus, $z = u + v$ with $u = (2t - 1)^4$ and $v = (2t + 1)^4$. Each of these will be treated as in the previous exercises. In particular, $u = x^4$ with $x = 2t - 1$. Then,

$$\frac{du}{dx} = 4x^3 = 4(2t - 1)^3 \text{ and } \frac{dx}{dt} = 2.$$

The chain rule gives

$$\frac{du}{dt} = 8(2t - 1)^3.$$

Similarly, $v = y^4$ with $y = 2t + 1$. Then,

$$\frac{dv}{dy} = 4y^3 = 4(2t+1)^3 \text{ and } \frac{dy}{dt} = 2.$$

The chain rule gives

$$\frac{dv}{dt} = 8(2t+1)^3.$$

Finally,

$$\frac{dz}{dt} = \frac{du}{dt} + \frac{dv}{dt} = 8(2t-1)^3 + 8(2t+1)^3 = 8\big((2t-1)^3 + (2t+1)^3\big). \tag{$*$}$$

As a check, we can expand z to get $z = 32t^4 + 48t^2 + 2$, so that

$$\frac{dz}{dt} = 128t^3 + 96t = 8(16t^3 + 12t).$$

The expansion of the expression for the derivative on line $(*)$ also gives this value.

Exercise 45.

$$z = \frac{\sqrt{t+1}}{\sqrt{t^2+1}}.$$

Since this is written as a quotient, let's try the quotient rule. This requires the derivatives of numerator and denominator. By the general power rule,

$$\frac{d}{dt}\sqrt{t+1} = \frac{1}{2}(t+1)^{-1/2} = \frac{1}{2\sqrt{t+1}}$$

and

$$\frac{d}{dt}\sqrt{t^2+1} = \frac{1}{2}(t^2+1)^{-1/2}(2t) = \frac{t}{\sqrt{t^2+1}}$$

Thus,

$$\frac{dz}{dt} = \frac{\sqrt{t^2+1}\frac{1}{2\sqrt{t+1}} - \sqrt{t+1}\frac{t}{\sqrt{t^2+1}}}{\left(\sqrt{t^2+1}\right)^2}$$

The denominator of this expression is just $t^2 + 1$. The numerator, which we shall call N while working with it, is a combination of fractions. Putting these fractions over a common denominator of $2\sqrt{t+1}\sqrt{t^2+1}$ gives

$$N = \frac{(t^2+1) - 2t(t+1)}{2\sqrt{t+1}\sqrt{t^2+1}} = \frac{1 - 2t - t^2}{2\sqrt{t+1}\sqrt{t^2+1}}.$$

Dividing by the denominator that was set aside while simplifying the numerator, we obtain

$$\frac{dz}{dt} = \frac{1 - 2t - t^2}{2(t+1)^{1/2}(t^2+1)^{3/2}}.$$

When the expressions in the numerator and denominator of a fraction are themselves powers, it is usually better to make the computation follow our **proof** of the quotient rule from the product rule instead of using the quotient rule statement, since the application of the quotient rule leads to an expression which is far from simple, and the simplification of that expression is no easier than mimicking the proof of the quotient rule.

This approach starts with

$$z = \left((t+1)^{1/2}\right)\left((t^2+1)^{-1/2}\right).$$

and uses the product rule with the derivatives of the factors found by the **general power rule**. Only a summary is given here; you should check that you understand how each term arises from these rules.

$$\begin{aligned}\frac{dz}{dt} &= \left((t+1)^{1/2}\right)(-t)\left((t^2+1)^{-3/2}\right) + \left((t^2+1)^{1/2}\right)\frac{1}{2}\left((t+1)^{-1/2}\right) \\ &= \left((t+1)^{-1/2}\right)\left((t^2+1)^{-3/2}\right)\left((-t)(t+1) + \frac{1}{2}(t^2+1)\right).\end{aligned}$$

Each term contains powers of $t+1$ and t^2+1. To combine these terms a common factor is chosen that is the product of **the smallest powers** of each of these expressions appearing in a term.

The last factor simplifies to $(1-2t-t^2)/2$, as before.

A third approach is to write $z = w^{1/2}$ with $w = (t+1)/(t^2+1)$. Then

$$\frac{dz}{dw} = \frac{1}{2}w^{-1/2} = \sqrt{\frac{t^2+1}{t+1}}$$

and

$$\frac{dw}{dt} = \frac{(t^2+1)(1) - (t+1)(2t)}{(t^2+1)^2}$$

You should check that all three approaches give the same function. You should also decide which approach you find most comfortable so that you will be prepared for similar questions on exams.

Exercise 49 Here the problem is stated in terms of an explicit composition, and you are to find all derivatives. Given

$$y = u^{-2/3} \text{ and } u = 2x^3 - x + 1,$$

we have

$$\frac{dy}{du} = \frac{-2}{3}u^{-5/3} \text{ and } \frac{du}{dx} = 6x^2 - 1.$$

In applying the chain rule, the u in dy/du must be expressed in terms of x, so

$$\frac{dz}{dt} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{-2}{3}(2x^3 - x + 1)^{-5/3}(6x^2 - 1).$$