Quotient Rule Exercises from Section 3.2

The quotient rule will be used in the form

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Exercise 17.

$$\begin{aligned}\frac{d}{dx}\left(\frac{x-1}{2x+1}\right) &= \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)^2}\\ &= \frac{2x+1-2x+2}{(2x+1)^2}\\ &= \frac{3}{2x+1^2}\end{aligned}$$

Note that the denominator is left in factored form. Although you are **not required to simplify answers**, and **mistakes in simplification can only hurt your grade**, it is natural to simplify results that will be used for further work. Also, if you know that an expression has a particular simplified form, it is a sign that a mistake has been made if you cannot simplify your answer to that form. In particular, a quotient of linear expressions always has a derivative that is a **constant** divided by the square of the original denominator. An alternate approach to Exercise 17 indicates why this is true.

First, rewrite the given expression.

$$\begin{aligned}\left(\frac{x-1}{2x+1}\right) &= \frac{1}{2}\cdot\frac{x-1}{x+{}^1\!/_2} = \frac{1}{2}\left(1-\frac{3}{2}\cdot\frac{1}{x+{}^1\!/_2}\right)\\ &= \frac{1}{2}-\frac{3}{4}\left(x+{}^1\!/_2\right)^{-1}\end{aligned}$$

The derivative of the constant ${}^1\!/_2$, that is the first term in this expression, is zero. The second term can be differentiated by the **general power rule** to get ${}^3\!/_4\left(x+{}^1\!/_2\right)^{-2}$. Note that the coefficient is a product of **three** factors: the original coefficient ${}^{-3}\!/_4$, the original exponent -1, and the chain rule factor 1 coming from differentiating the base $x+{}^1\!/_2$.

Exercise 27. If

$$f(x) = \frac{(x+1)(x^2+1)}{x-2},$$

then

$$f'(x) = \frac{(x-2)\frac{d}{dx}\big((x+1)(x^2+1)\big) - (x+1)(x^2+1)(1)}{(x-2)^2}.$$

To continue, we need

$$\frac{d}{dx}\big((x+1)(x^2+1)\big) = (x+1)(2x) + (x^2+1)(1) = 2x^2+2x+x^2+1 = 3x^2+2x+1,$$

which could also be easily obtained by expanding $(x+1)(x^2+1)$ and differentiating term-by-term. Thus,

$$\begin{aligned}f'(x) &= \frac{(x-2)(3x^2+2x+1)-(x+1)(x^2+1)(1)}{(x-2)^2}\\ &= \frac{3x^3-4x^2-3x-2-x^3-x^2-x-1}{(x-2)^2} = \frac{2x^3-5x^2-4x-3}{(x-2)^2}\end{aligned}$$

Exercise 29. Sometimes the difficult rules are applied **after** the simple ones. In particular, the straightforward nature of the **sum rule** means that two different differentiation problems can be considered as one simply by asking to differentiate the sum (or difference) of the two functions. The additional complexity of combining the functions into one in this way should be negligible. To differentiate

$$f(x) = \frac{x}{x^2 - 4} - \frac{x-1}{x^2+4},$$

one should write $f(x) = r - s$ (the names of the terms are **completely arbitrary**) with

$$r = \frac{x}{x^2-4} \quad \text{and} \quad s = \frac{x-1}{x^2+4}.$$

The quotient rule in each of these separate cases gives

$$\frac{dr}{dx} = \frac{(x^2-4)(1) - (x)(2x)}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}$$
$$\frac{ds}{dx} = \frac{(x^2+4)(1) - (x-1)(2x)}{(x^2+4)^2} = \frac{-x^2+2x+4}{(x^2+4)^2}$$

Combining terms, **with no further simplification**,

$$f'(x) = \frac{-x^2-4}{(x^2-4)^2} - \frac{-x^2+2x+4}{(x^2+4)^2}.$$

Exercise 33. A favorite exercise is one in which only the values of functions and their derivatives at special points are known, so that the **differentiation rules** are to be applied to **unknown functions** and the results interpreted using **known values**. This example is more elaborate than most.

Given f and g known to be differentiable at $x = 1$ with

$$f(1) = 2 \qquad g(1) = -2$$
$$f'(1) = -1 \qquad g'(1) = 3$$

and

$$h(x) = \frac{xf(x)}{x + g(x)}$$

find $h'(1)$.

The formulas giving the derivatives of the numerator and denominator of the expression defining $h(x)$ are easy applications of simpler rules, so they will be inserted as needed as the quotient rule is applied.

$$h'(x) = \frac{x + g(x)\, f(x) + xf'(x) - xf(x) 1 + g'(x)}{(x+g(x))^2}$$

Algebraic work with this expression is not likely to make evaluation at $x = 1$ any easier, so we work directly with this expression, which contains only $f(x)$, $g(x)$, $f'(x)$,g'(x), and some explicit functions of x, all of which are known at $x = 1$. Thus

$$h'(1) = \frac{\big(1+(-2)\big)\big(2+(1)(-1)\big) - (1)(2)\big(1+3\big)}{\big(1+(-2)\big)^2}$$
$$= \frac{(-1)(1)-(2)(4)}{(-1)^2} = \frac{-1-8}{(1)} = -9$$

Section 3.5 Most of the problems here ask for first and second derivatives. It is useful to simplify the first derivative to have a clearer view of how to find its derivative.

Exercise 2. Starting from a polynomial, answers can be written as fast as the given expression can be read.

$$f(x) = -0.2x^2 + 0.3x + 4$$
$$f'(x) = -0.4x + 0.3$$
$$f''(x) = -0.4$$

All positive degrees are decreased by one in differentiation, and constant terms have derivative zero, so **successive derivatives of polynomials keep getting simpler**. In this example, $f^{(n)}(x) = 0$ for all $n > 2$.

Exercise 12. Let $h(w) = (w^2+2w+4)^{5/2}$. Thus, $z = h(w) = v^{5/2}$ with $v = w^2 + 2w + 4$. Then,

$$h'(w) = \frac{5}{2}(w^2+2w+4)^{3/2}(2w+2).$$

Here the colors identify the two factors dz/dv and dv/dw. Application of the **chain rule** here gives a derivative that is a products, so the **product rule** (as well as the chain rule) will be needed when finding the second derivative. To prepare $h'(w)$ to be differentiated, the constant factor can be combined with the linear factor to get

$$h'(w) = (5w+5)\cdot(w^2+2w+4)^{3/2}.$$

The application of the product rule will require a formula similar to that used in finding $h'(w)$ which can be developed and recorded **before** it is needed:

$$\left(\frac{d}{dw}\right)(w^2+2w+4)^{3/2} = \frac{3}{2}(w^2+2w+4)^{1/2}(2w+2) = h'(w) = (3w+3)\cdot(w^2+2w+4)^{1/2}.$$

Now,

$$h''(w) = (5w+5)(3w+3)\cdot(w^2+2w+4)^{1/2} + (w^2+2w+4)^{3/2}(5).$$

If you need to simplify this, a common factor of $(w^2+2w+4)^{1/2}$ can be removed, writing

$$(w^2+2w+4)^{3/2} = (w^2+2w+4)^{1/2}\cdot(w^2+2w+4).$$

Thus,

$$\begin{aligned} h''(w) &= (w^2+2w+4)^{1/2} \cdot \big((5w+5)(3w+3) + (w^2+2w+4)(5) \big) \\ &= (w^2+2w+4)^{1/2} \cdot (15w^2+30w+15+5w^2+10w+20) \\ &= (w^2+2w+4)^{1/2} \cdot (20w^2+40w+35) \end{aligned}$$

Exercise 16. Let

$$g(t) = \frac{t^2}{t-1},$$

and differentiate using the quotient rule to obtain

$$\begin{aligned} g'(t) &= \frac{(t-1)(2t)+(t^2)(1)}{(t-1)^2} \\ &= \frac{t^2-2t}{(t-1)^2} \\ &= (t^2-2t)\cdot(t-1)^{-2} \end{aligned}$$

Negative exponents are used here so that differentiation by the **product rule** will directly give an expression with a factor of $(t-1)^{-3}$. This is **sure to be simpler** than the result of the **quotient rule**, which will have $(t-1)^4$ in the denominator. Continuing,

$$\begin{aligned} g''(t) &= (t^2-2t)(-2)(t-1)^{-3}(1) + ((t-1)^{-2})(2t-2) \\ &= (t-1)^{-3}\cdot(-2t^2+4t+2t^2-4t+2) \\ &= 2(t-1)^{-3} \end{aligned}$$