Elasticity of demand (3.4)

Given a relation between **price** p and **inventory** x, the quantity E is defined by

$$E = -\frac{p}{x}\frac{dx}{dp}.$$

If $E > 1$, one says **demand is elastic**, which is supposed to be a **good thing**. It corresponds to a situation in which a **decrease in price** leads to an **increase in revenue**. Questions often are phrased as: is demand elastic?

Exercise 22

Given

$$x = -\frac{3}{2}p + 9$$

$$\frac{dx}{dp} = -\frac{3}{2}$$

$$E = \frac{\frac{3}{2}p}{9 - \frac{3}{2}p}$$

To test elasticity at $p = 2$. the expression giving E in terms of p can be evaluated to get $E = 1/2$ at $p = 2$, so demand is **inelastic**.

Note that an equation for a **demand curve** has a reasonable interpretation in economics only where p and x are positive. Unfortunately, the nicest algebraic expression satisfy this only over a limit interval of values. For this example, x is positive only for $p < 6$. This restriction should be included in any work with this model. Under this assumption, $E > 1$ if and only if $3/2\,p > 9 - 3/2\,p$, which simplifies to $p > 3$.

Furthermore, since $R = px = p\left(9 - 3/2\,p\right) = 9p - 3/2\,p^2$, it can also be seen directly that R is an increasing function of p (the inelastic case) if $p < 3$ and R is a decreasing function of p (the elastic case) if $p > 3$.

Exercise 26

Given $p = 144 - x^2$, for which $0 \le x \le 12$ should be assumed to have a meaningful model, we can solve for x to get $x = \sqrt{144 - p}$. Differentiating, $dx/dp = (1/2)(144 - p)^{-1/2}(-1)$. Since x **is a synonym for** $\sqrt{144 - p}$, we could write $dx/dp = -1/(2x)$, which will be **directly** available when we have implicit differentiation (i.e., almost immediately).

Now, the given value of $p = 96$ leads to $x = \sqrt{48}$ and $E = 1$, which is called **unitary demand**, the state separating the elastic and inelastic states.

Implicit differentiation (3.6)

In response to a request for a **formula** for implicit differentiation, if was noted that these problems are done by following a **method** that is easily described **without being expressed by a formula**. Here is the description of that method: **Given an equation** containing x and y, **assume** that y is a **function** of x that **satisfies the equation identically**. Then, **differentiate** with respect to x and **solve** for dy/dx. Of course, other variables may be used in place of x and y.

Exercise 10

Given $2x^2 + y^2 = 16$, differentiate to obtain

$$4x + 2y\frac{dy}{dx} = 0.$$

Solve to obtain

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

That's all there is to it! The expression for dy/dx depends on both x and y. In this example, the curve is an ellipse. This curve is symmetrical with respect to reflecting in the coordinate axes, but such reflections take lines of slope m to lines of slope $-m$, and the formula for dy/dx clearly has this property.

Exercise 17 Given $x^{1/2} + y^{1/2} = 1$, differentiate to obtain

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0.$$

Solve to obtain

$$\frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}.$$

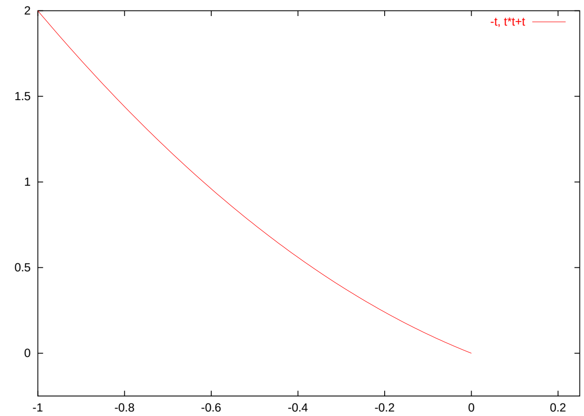
Exercise 19 Given $\sqrt{x+y} + x = 0$, differentiate to obtain

$$\frac{1}{2}(x+y)^{-1/2}\left(1 + \frac{dy}{dx}\right) + 1 = 0.$$

Solving:

$$1 + \frac{dy}{dx} = -2\sqrt{x+y}$$
$$\frac{dy}{dx} = -1 - 2\sqrt{x+y}$$

Since $\sqrt{x+y}$ always means the **positive** square root of $x+y$, this equation represents the portion of the curve $x + y = x^2$ with $x \leq 0$. Here is a sketch:



In agreement with the fact that the expression obtained for dy/dx is always negative, this curve is seen to give y as a decreasing function of x.

An algebraic solution of given equation gives $y = x^2 - x$, which is easily recognized as a parabola. However, the portion of the curve with $x > 0$ is an **extraneous solution** of the given equation. The ability to solve for y shows that the dy/dx should be $2x - 1$, and a glance at the given equation and the result of implicit differentiation shows that **it is**.

Related rates (3.6) Work with this topic is similar to implicit differentiation in that an **identical relation** between quantities is differentiated to get a new relation that includes the derivatives of those quantities. A difference between these two types of problems is that, often in a Related Rates problem, the independent variable **does not appear** in the relation.

Exercise 40 This exercise deals with two cars leaving an intersection, one heading West and one heading North. All statements about **time** are given in **seconds**; all statements about **distance** are given in **feet**; and in a great triumph of consistency, all statements about **velocity** are given in **feet per second**. If these units are used, then the values of these quantities **may be treated as numbers**. Then t will be used for time, x for the distance of the first car West of the intersection, and y for the distance of the second car North of the intersection. The exercise asks about the **straight line distance** between the cars, and we denote this by z (also measured in feet). Pythagoras tells us that

$$x^2 + y^2 = z^2,$$

so that the derivative of Pythagoras tells us that

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At the time described in the exercise (which is $t = 4$, although no explicit use will be made of this value) we are given

$$x = 20 \qquad y = 28$$
$$\frac{dx}{dt} = 9 \qquad \frac{dy}{dt} = 11$$

First, use Pythagoras to get $z^2 = 20^2 + 28^2 = 1184$, so that $z = \sqrt{1184} = 4\sqrt{74}$ at this value of t. Then, put all the numerical values at this time into the derivative of Pythagoras to get $8\sqrt{74}(dz/dt) = 2(20)(9) + 2(28)(11) = 976$. Solving, $dz/dt = 122/\sqrt{74} \approx 14.1822$.