Section 3.7: Differentials The simplest type of exercise expresses formulas for derivatives in the **notation** of differentials. Other exercises use points on an **easily found** tangent line to approximate points on a curve.

Exercise 1 If $y = x^2$, then $dy = (4x)dx$. While that brief statement is a complete solution to the exercise, there are many interpretations that can be added, usually involving treating the expressions dx and dy as **new variables**.

Exercise 15 If $y = x^2 - 1$, then $dy = (2x)dx$. The phrase "x changes from 1 to 1.02" is expressed by setting $x = 1$ (the **base value**) and $dx = 0.02$ (the change, so that $x + dx = 1.02$). Using these values in the formulas, $y = (1)^2 - 1 = 0$, and $dy = 2(1)(0.02) = 0.04$. Thus y changes from 0 to **approximately 0.04**.

The expression $x^2 - 1$ is simple enough to evaluate **exactly** at $x = 1.02$, so we find that the value of y at that point is 0.0404. It thus appears that the approximation given by the differential accounts for most of the change in the function between these nearby points.

At this stage in the development of calculus, it is only possible to give examples of approximations. For an approximation to be useful, you need a **quantitative** measure of how good the approximation is. Unfortunately, this depends on estimates that are beyond the scope of the calculus we have at this point in the course. This means that it may not be easy to describe what has been accomplished, even if the everything requested in the exercise has been done.

Exercise 17 If $y = 1/x$, then $dy = (-1/x^2)dx$. The phrase "x changes from -1 to -0.95" is expressed by setting $x = -1$ (the **base value**) and $dx = 0.05$ (the change, so that $x + dx = -0.95$). Using these values in the formulas, $y = 1/(-1) = -1$, and $dy = (-1/(-1)^2)(0.05) = -0.05$. Thus y changes from -1 to **approximately -1.05**.

My calculator has no difficulty finding $1/(-0.95) = -1.05263157895$. Differentials have given the first two digits after the decimal point, but there is a clear difference in the **third** decimal place.

Exercise 20 In the world without calculators (which wasn't all that long ago), if you met a number like $\sqrt{17}$, you wanted to have some idea of its value in order to check whether this number made numerical sense in the context that led to its calculation. To do this, you would **invent** something like the exercises we just did leading to an expression of the form $y + dy$ that could be treated as an approximation to $\sqrt{17}$.

The most reasonable function to use is $y = \sqrt{x} = x^{1/2}$, so that $dy = \frac{1}{2}x^{-1/2}dx$. We take $x = 16$ to get $y = 4$. Then $dx = 1$ and $dy = (1/2)(16)^{-1/2}(1) = (1/2)(1/4)(1) = \frac{1}{8}$. Thus $\sqrt{17} \approx 4 + \frac{1}{8} = 33/8$. This differs from the exact value by less than 0.002.

Exercise 24 To estimate $\sqrt[4]{81.6}$, use $y = x^{1/4}$, so that $dy = \frac{1}{4}x^{-3/4}dx$. Take $x = 81$ to get $y = 3$ and $dx = 0.6$ to get

$$dy = \frac{1}{4}(81)^{-3/4}(0.6) = \frac{1}{4}\frac{1}{27}(0.6) = 0.00555\ldots .$$

Thus, differentials give $\sqrt[4]{81.6} \approx 3.00555556$, while the true value is $3.0055401\ldots$, giving an error less than 1.6×10^{-5}.

Exercise 30 A cube of edge 30 cm is to be covered with paint to a thickness of 0.05 cm. The resulting cube will have an edge of 30.1 cm (since paint has been applied to both ends of the edge). The volume V of a cube of edge x is given by $V = x^3$, so $dV = 3x^2 dx$. If distances are measured in cm, this gives $x = 30$ and $dx = 0.1$. At these values $V = 27000$ and $dV = 3(30)^2(0.1) = 270$. This predicts that about 270 cm^3 of paint would be needed. A more careful analysis reveals the this corresponds to covering each face to the given thickness, ignoring the bead of paint surrounding each edge. In particular, this shows that this estimate is a little too small.