Old Business: Section 3.6

Some slides have been found that were not previously transcribed.

Exercise 17 This asked for dy/dx by implicit differentiation with y defined as a function of x by $x^{1/2} + y^{1/2} = 1$. In the summary for lecture 9, this was easily found and simplified to

$$\frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}.$$

Other exercises in Section 3.6 asked for **second derivatives** of functions defined implicitly. As with all second derivatives, this should be found by differentiating dy/dx, maintaining the assumption that y is a function of x. This gives

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{1}{2}\left(\frac{y}{x}\right)^{-1/2}\frac{d}{dx}\left(\frac{y}{x}\right) \\
&= -\frac{1}{2}\left(\frac{y}{x}\right)^{-1/2}\frac{x\frac{dy}{dx} - y(1)}{x^2} \\
&= \frac{1}{2}x^{-3/2}y^{-1/2}\left(x^{1/2}y^{1/2} + y\right) \\
&= \frac{1}{2}x^{-3/2}\left(x^{1/2} + y^{1/2}\right) = \frac{1}{2}x^{-3/2}
\end{aligned}$$

This result is so **strikingly simple** that it **demands an independent verification**. This can be done by finding an **explicit** expression for y in terms of x before doing any calculus. The algebra is

$$\begin{gathered}
y^{1/2} = 1 - x^{1/2} \\
y = (1 - x^{1/2})^2 = 1 - 2x^{1/2} + x \\
\frac{dy}{dx} = -x^{-1/2} + 1 \\
\frac{d^2y}{dx^2} = \frac{1}{2}x^{-3/2}
\end{gathered}$$

as obtained previously. Along the way, an explicit expression for dy/dx was found that is also easily seen to be equivalent to the implicit expression obtained earlier.

We also note that the notation of differentials can be used to make the calculation of the derivative of y/x **appear** simpler. Instead of being forced to declare x as the independent variable, the quotient rule can be written in terms of differentials as

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2},$$

and the result of implicit differentiation used in the form

$$x^{1/2}\,dy + y^{1/2}\,dx = 0.$$

Exercise 41 The variables x, representing the **weekly demand** in thousands for Super Titan radial tires, and p, representing the **unit price** in dollars for the tires, are related by the **demand equation**

$$p + x^2 = 144.$$

At a certain time, $x = 9$ and $p = 63$. (This is consistent with the given demand equation because $63 + 9^2 = 63 + 81 = 144$.) It is also given that the price per tire is increasing at \$2 per week, which can be written as $dp/dt = 2$, using the given scale for p and measuring time in weeks. The derivative of the given demand equation is

$$\frac{dp}{dt} + 2x\frac{dx}{dt} = 0.$$

Substituting the given values for p, x, and dp/dt gives

$$2 + 2(9)\frac{dx}{dt} = 0.$$

Solving gives $dx/dt = -1/9$. That is, the demand is **decreasing** at the rate of $1000/9$ tires per week per week. (The first phrase represents the units of x, re-expressed at individuals rather than thousands; the second phrase represents the units of t. This may be interpreted as saying that in the week of the price increase, demand drops from 9000 tires per week to 8889 tires per week.)

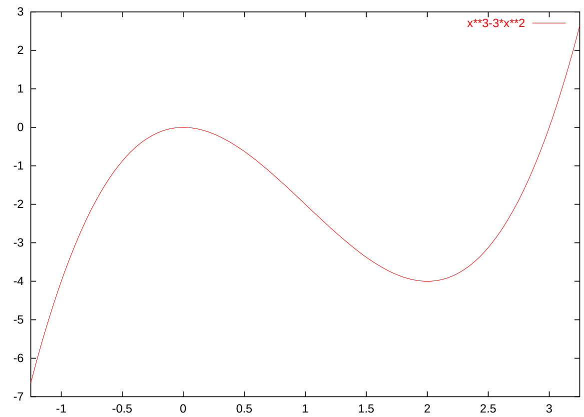
Section 4.1 Exercise 4 refers to a graph given in the textbook, with no expression for the function being graphed. In the other exercises, an expression is given. If appropriate, graphs will be added to illustrate our conclusions. General instructions are to find where the function is increasing and where it is decreasing. Where appropriate, relative maxima and minima will be noted even if not requested in the statement of the exercise. Three of the exercises deal with polynomials of degree 3. The discussion will seem repetitious, since the qualitative aspects of these graphs are identical in the sense that they differ only in changes of scale and origin on both axes. A minor variation is possible: if the coefficient of x^3 is negative, the curve will resemble a **reflection** of the graph shown for exercise 14. Another variation is possible, but somewhat dull: $f'(x)$ could not factor into distinct real linear factors, but then $f(x)$ would have no critical points, so would be either everywhere increasing or everywhere decreasing.

Exercise 4 A graph is given, and you are asked where it is increasing and where it is decreasing. Since the function is unbounded near $x = 0$, it should be undefined at zero. One sees that the $f(-1) = -2$ is greater than all other values of $f(x)$ with $x < 0$, with the function increasing for $x < -1$ and decreasing for $-1 < x < 0$. Here, the function increases to a relative maximum at $x = -1$, and then decreases. For positive x, $f(x)$ is deceasing for $0 < x < 1$, reaching a relative minimum of $f(1) = 2$, then increasing for $x > 1$.

If one were to fail to mention that $x = 0$ is excluded, the description would sound paradoxical: $f(x)$ decreases from $x = -1$ to $x = 1$, yet $f(-1) = -2 < 2 = f(1)$. A simpler function illustrating the same paradox is $g(x) = 1/x$ whose derivative $g'(x) = -1/x^2$ is negative wherever it is defined, so it is decreasing on every interval in its domain, yet the value at any positive x is greater than the value at any negative x.

As long as you have a picture, there should be no confusion. However, calculus allows many properties of functions to be determined without reference to a graph. When this is done, it is important to notice if any points are excluded from the domain of the function.

Exercise 14 Given $f(x) = x^3 - 3x^2$, differentiation gives $f'(x) = 3x^2 - 6x = 3x(x-2)$. From the factored form of $f'(x)$, it follows that $f'(x) = 0$ for $x = 0$ and $x = 2$. If $x > 2$, both factors in the expression for $f'(x)$ are positive, so $f'(x) > 0$ and $f(x)$ is increasing. As x moves to the left on the number line, when it passes a point where $f'(x) = 0$, one of the factors in $f'(x)$ changes sign and the others retain their sign. Thus, for $0 < x < 2$, $f(x)$ is decreasing, and for $x < 0$, $f(x)$ is increasing. This shows that there is a **relative maximum** at $x = 0$ and $f(0) = 0$; and a **relative minimum** at $x = 2$ and $f(2) = -4$. Although these values are **relative extrema**, the function does take values outside this interval, as see in the following graph.



Exercise 16 Given $f(x) = x^3 - 3x + 4$, differentiation gives $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$. Now the critical points are $x = \pm 1$. As in the previous exercise, $f'(x)$ is negative for $-1 < x < 1$, so $f(x)$ is decreasing there. **Outside** this interval, $f(x)$ is increasing. The appearance of the graph is similar to the previous example, so it will not be shown.

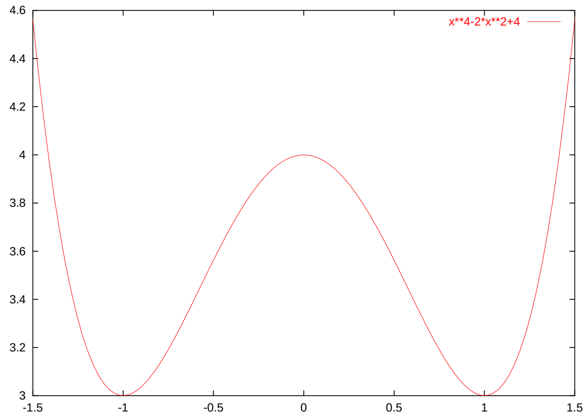
Exercise 18 Given

$$f(x) = \frac{2}{3}x^3 - 2x^2 - 6x - 2,$$

differentiation gives $f'(x) = 2x^2 - 4x - 6 = 2(x-3)(x+1)$. As in the previous exercises, $f'(x)$ is negative for $-1 < x < 3$, so $f(x)$ is decreasing there. **Outside** this interval, $f(x)$ is increasing. The appearance of the graph is similar to the previous example, so it will not be shown.

All of these examples of polynomials of degree 3 are similar. The remaining exercises involve a variety of different functions

Exercise 20 Given $g(x) = x^4 - 2x^2 + 4$, differentiation gives $g'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$. There are now three critical points: $x = -1$, $x = 0$, and $x = 1$. The factor of $g'(x)$ giving each of these is **simple** (i.e., appearing only to the first power), so $g'(x)$ **changes sign at each critical point, and nowhere else**. Since $g'(x)$ has a positive leading coefficient, it is positive for x to the right of all of its roots. Thus: $g'(x) > 0$ for $1 < x$; $g'(x) < 0$ for $0 < x < 1$; $g'(x) > 0$ for $-1 < x < 0$; $g'(x) < 0$ for $x < -1$. Hence, $g(x)$ changes from being decreasing to being increasing at $x = -1$ and at $x = 1$, giving **relative minima** at $(-1, 3)$ and $(1, 3)$; while $g(x)$ changes from being increasing to being decreasing at $x = 0$, so there is a **relative maximum** at $(0, 4)$. Here is the graph.



Exercise 24 Given

$$g(t) = \frac{2t}{t^2+1},$$

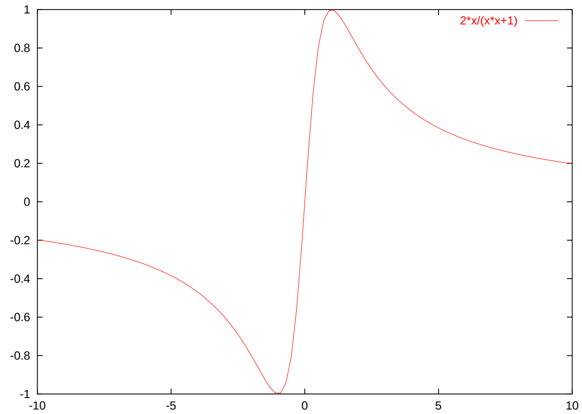
differentiation gives

$$\frac{(t^2+1)(2) - (2t)(2t)}{(t^2+1)^2} = \frac{2-2t^2}{(t^2+1)^2}.$$

The denominator of this expression is always positive, so we examine the numerator to find the sign of $g'(t)$. This numerator factors as $-2(t-1)(t+1)$, so it is negative for large t, becomes positive for $-1 < t < 1$, then negative again for $t < -1$. The function $g(t)$ is thus increasing only for $-1 < t < 1$, and otherwise decreasing. Since the denominator of $g(t)$ has larger degree than the numerator,

$$\lim_{t \to \infty} g(t) = 0.$$

There is a minimum at $(-1, -1)$ and a maximum at $(1, 1)$. The graph below also shows the symmetry about the origin characteristic of an **odd function**, i.e., a function that satisfies $g(-t) = -g(t)$.



Exercise 74 The height (in feet) attained by a rocket t seconds into flight is given by

$$h(t) = -\frac{1}{3}t^3 + 16t^2 + 33t + 10$$

Where is the rocket rising? Where is it falling?

The words serve mainly to introduce the function $h(t)$ and give a reason for being interested in where it is increasing. There is an **implicit assumption** that only those $t \geq 0$ for which $f(t) \geq 0$ are relevant (roughly $0 \leq t \leq 50$), but the solution of the problem proceeds in the same way as problems stated more directly. Differentiate to find

$$h'(t) = -t^2 + 32t + 33 = -(t+1)(t-33).$$

The expression defining $h(t)$ is increasing for $-1 < t < 33$ and decreasing otherwise. If only positive values of t are relevant, one can answer the question in the language used in its statement by saying that the rocket is rising for the first 33 second of its flight and falling after that. Some more details are that $f(33) = 6544$, so that the rocket reaches a maximum height of 6544 feet (a little less than a mile and a quarter), and hits the ground after approximately 48.992 seconds at a speed of 866.48 feet per second (or 590.78 miles per hour).

Exercise 83 The amount of nitrogen dioxide present in the atmosphere on a certain day in the city of Long Beach is approximated by

$$A(t) = \frac{136}{1 + 0.25(t-4.5)^2} + 28$$

for $0 \leq t \leq 11$, where t is the **time in hours** on that day with $t = 0$ corresponding to 7 AM. Thus $t = 11$ corresponds to 6 PM. Find where $A(t)$ is increasing and interpret your results.

This can be answered by finding where $A'(t) > 0$. Begin by writing

$$A(t) = 136\Big(1 + 0.25(t-4.5)^2\Big)^{-1} + 28$$

to allow use of a power rule instead of the **more complicated** quotient rule. Then

$$A'(t) = 136(-1)\Big(1 + 0.25(t-4.5)^2\Big)^{-2}\big(0.25(t-4.5)(1)\big)$$

where the first part of the expression is the derivative of $136u^{-1} + 28$ and the second part is the derivative of $u = 1 + 0.25(t - 4.5)^2$. Since the first part is a negative constant times a power of an expression that is a sum of squares, it is always negative. The second part is positive for $t > 4.5$ and negative for $t < 4.5$. Combining these two factors, we find that A increases for $0 < t < 4.5$ and then decreases over the rest of the domain. In the language of the model, this measure of pollution increases from 7 AM to 11:30 AM, and decreases for the rest of the day.

Note that our interpretation of the chain rule gave that $A(t)$ was increasing where $1+0.25(t-4.5)^2$ was decreasing because of the simple dependence of $A(t)$ on this quantity. The ability to calculate derivatives easily allows us to notice this **after calculating the derivative** instead of **requiring this observation** as part of the analysis of the problem. Here is a graph of $A(t)$