Section 4.3

This lecture included several worked exercises in the prepared slides since the exercises dealt with graph sketching and crude sketches would not be very illuminating.

The graphs from exercises 29 and 30 of the text were projected (in color). These exercises showed graphs of a function and its derivative and asked to identify which was which. The key is that a function is increasing **exactly** where its derivative is positive. In particular, the derivative will change sign at critical points of the primitive function.

In #29, $g(x)$ is seen to be zero near $x = -2$ while $f(x)$ is rapidly increasing there. This **eliminates** the possibility the $g(x)$ could be the derivative of $f(x)$. You can then check the contrary case that $f(x)$ is the derivative of $g(x)$. Look at the intervals where $f(x)$ has a fixed sign: it is negative for $x < -3/2$ and again for $0 < x < 3/2$. Concentrating on these intervals for the graph of $g(x)$, we see that the graph is decreasing there, and that the endpoints of the intervals give critical points of $g(x)$. There are no other critical points and $g(x)$ is increasing where $f(x) > 0$.

In #30, $g(x) > 0$ for all $x > -0.8$ (approximately), but $f(x)$ decreases until $x = 1$. Again, $g(x)$ **cannot be** the derivative of $f(x)$. The contrary possibility that $f(x)$ is the derivative of $g(x)$ is supported by the appearance that the interval where $f(x) < 0$ coincides with where $g(x)$ is decreasing.

The graphs on the prepared slides included tangent lines at selected points of the curves. The were easy to include by writing the tangent line to $y = f(x)$ at $x = a$ in the form

$$y = f'(a)(x - a) + f(a).$$

Since this has the form of the graph of a function, the graphing program can draw it easily. It is not necessary to modify the algebraic form of the equation of this line before graphing it. **On the contrary**, this form of the equation is better for computing in the interval of interest than other forms since $x - a$ will be small in the graphing window.

In calculus, the **tangent line to a curve** is always found in this way, since this is the **definition** of that line. At a point of inflection, the tangent line defined in this way will **cross the curve**, unlike the behavior of tangents to **convex** curves like circles. The connotation of “touching, but not cutting” the curve that distinguishes tangents of circles has been replaced by more sophisticated (and more useful) idea of being “the best approximating line” to the curve. Since the slope of the tangent line at $x = a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

we have

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

The parenthesized expression may be written as

$$\frac{f(x) - \big(f(a) + f'(a)(x - a) \big)}{x - a}.$$

To say that this approaches zero say that the difference between $f(x)$ and the expression for y on the tangent line at $x = a$ is **much smaller than** $x - a$. On all other lines through $\big(x, f(a)\big)$ (except the vertical line), this distance will be **comparable to** $x - a$.

A more careful analysis of this quantity shows that it is typically **smaller** for tangent lines at inflection points than it is for other tangent lines. The final graph in the prepared slides illustrates this. The tangent line remains close to the curve throughout the graphing window, while a tangent line at points near the right side of the figure would only touch the curve briefly and then move relatively for from the curve.