Section 4.4 Exercise S-C. 3

We are given a function

$$f(t) = 80 + \frac{1200t}{t^2 + 40000}$$

for $0 \le t \le 250$. Consult the text for the description of the model leading to this function and the words that signify that we want to find the **maximum value on this interval**. The general theory says that we need to check the value of $f(t)$ at $t = 0$, $t = 250$, and **at any critical points between** 0 **and** 250. To find critical points, determine

$$f'(x) = \frac{(t^2 + 40000)(1200) - (1200t)(2t)}{(t^2 + 40000)^2} = \frac{48000000 - 1200t^2}{(t^2 + 40000)^2}$$

The denominator is always positive, so our formula for $f'(t)$ is defined everywhere. To find the critical points, it is only necessary to consider the numerator. The condition for critical points is

$$48000000 - 1200t^2 = 0$$
$$1200t^2 = 48000000$$
$$t^2 = 40000$$
$$t = 200 \text{ or } t = -200$$

Only the root $t = 200$ need be considered further since it is the only one in the **given** domain of $f(t)$. The expression for $f'(t)$ shows us that $f(t)$ increases for $0 < t < 200$ and then decreases, so the value at this critical point will be a maximum. However, the values at the endpoints **should be computed** because it is easy to do so and it provides a check on the work of finding the derivative and solving the equation that identifies where it is zero.

We find that $f(0) = 80$, $f(200) = 83$ and $f(250) = 3400/41 \approx 82.93$

Exercise 8

Consider the graph given in the textbook. No formula is given for this function $f(x)$ — only the graph and the information that the domain is the interval $[-1, 3]$. The picture shows that $f(-1) = -1$ and $f(3) = +1$. There is also one point where the graph has a **horizontal tangent**, which marks a **critical point** at $x = 0$. Interpreting the graph, it looks like $f(0) = -3$. The theorem on maxima and minima says that the extreme values of the function are taken only at an endpoint or a critical point. This restricts our search for these values to

$$f(-1) = -1 \qquad f(3) = +1 \qquad f(0) = -3.$$

Sorting the **function values** shows that the minimum is $f(0) = -3$ and the maximum is $f(3) = +1$.

Exercise 17

We are given the function

$$f(x) = -x^2 + 4x + 6,$$

and the **domain** $[0, 5]$. The endpoints that must be considered and the values of f at those points are $f(0) = 6$ and $f(5) = 1$. Now we look for **critical points**. Differentiating gives

$$f'(x) = -2x + 4,$$

which is zero only for $x = 2$. Since $2 \in [0, 5]$ (i.e., $0 \le 2 \le 5$), we need to also consider $f(2) = 10$. This replaces $f(0) = 6$ as the largest value of the function, but leaves $f(5) = 1$ as the smallest value.

Exercise 18 This uses the same expression for $f(x)$ as the previous exercise, but **changes the domain** to $[3, 6]$. Since **only** $x = 2$ can be a critical point (this is determined by the expression for $f(x)$), and $2 \notin [3, 6]$, only the endpoints need be considered. We find $f(3) = 9$ and $f(6) = -6$, the first of these is the maximum and the second is the minimum. Note that our expression for $f'(x)$ shows that $f'(x) < 0$ **on the whole domain**, so that $f(x)$ is decreasing. This agrees with our observation that the maximum is at the left end of the domain.

Exercise 37 Now

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}$$

on $[-1, 1]$. To Compute the function at the endpoints, we first note that

$$\sqrt{x^2 + 1} = \sqrt{2}$$

at both endpoints, so $f(-1) = -1/\sqrt{2}$ and $f(1) = 1/\sqrt{2}$. Now, we look for critical points. Before starting the differentiation, we rewrite

$$f(x) = x(x^2 + 1)^{-1/2}$$

so that we can use the (easier) product rule instead of the quotient rule. We also need

$$\frac{d}{dx}(x^2 + 1)^{-1/2} = -\frac{1}{2}(x^2 + 1)^{-3/2}(2x) = -x(x^2 + 1)^{-3/2}.$$

By isolating this, it is possible to differentiate and simplify this quantity before combining it with the other quantities appearing in the product rule. Now

$$f'(x) = x\left(-x(x^2 + 1)^{-3/2}\right) + (x^2 + 1)^{-1/2}(1).$$

Both terms in this expressions contain powers of $(x^2 + 1)$. The exponents are $-3/2$ and $-1/2$. If we use the **lower** power as a common factor, we consider

$$(x^2 + 1)^{-1/2} = (x^2 + 1)(x^2 + 1)^{-3/2},$$

allowing a **polynomial** to remain when this factor is extracted. Thus,

$$f'(x) = (x^2 + 1)^{-3/2}(-x^2 + x^2 + 1) = (x^2 + 1)^{-3/2}.$$

This quantity is always positive, so there are no critical points and the extreme values are the values at the endpoints that were found earlier.

Exercise 46 At the end of a long story, you are asked to **maximize profit**. The first requirement for successful solution of such problems is **patience**. To maximize a function, you must first **have** the function. Here, **some assembly is required**. You are not given an expression for profit directly, but you have information about **cost** and the **demand equation** relating price to inventory, from which you can compute **revenue**. Then,

$$\text{Profit} = \text{Revenue} - \text{Cost}.$$

All quantities are given in terms of consistent units with money measured in dollars and time in days. The independent variable x represents the number of tennis rackets made per day. The demand equation is

$$p = 10 - 0.0004x,$$

where p, as usual, represents **price** in dollars per tennis racket. Thus

$$R = px = 10x - 0.0004x^2,$$

where R is **Revenue** in dollars per day. We were also given

$$C = 400 + 4x + 0.0001x^2,$$

where C is the cost in dollars per day of making x tennis rackets per day. Now

$$P = R - C = -400 + 6x - 0.0005x^2$$

is an expression for profit. The hard work has been done and we are ready to apply calculus by looking for critical points. This requires computing

$$\frac{dP}{dx} = 6 - 0.001x,$$

and doing the algebra to show that this is zero when $x = 6000$. We also note that dP/dx is positive for smaller values of x and negative for larger values of x, so this is the maximum value we seek. The question asked for a **daily level of production**, so the answer is 6000 rackets/day.