Section 4.5

Here are the solutions of the problems given in the prepared slides.

Exercise 4 We repeat the statement of the problem: "By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, an open box may be made. If the cardboard is 15 in. long and 8 in. wide, find the dimensions of the box that will yield the maximum volume."

The discussion in the prepared slides used s, the side of the removed square in inches, as the independent variable and gave the volume of the box in cubic inches by the formula

$$V = (15 - 2s)(8 - 2s)s = 120s - 46s^2 + 4s^3$$

We note that $V = 0$ at the endpoints $s = 0$ and $s = 4$, and that $V > 0$ inside the interval. This assures us that the maximum will be assumed at an interior critical point. The critical point was found from

$$\frac{dV}{ds} = 120 - 92s + 12s^2 = 4(6 - s)(5 - 3s)$$

The only critical point in the **interior** of the interval is $s = 5/3$. At this point,

$$V = (15 - 10/3)(8 - 10/3)(5/3) = (35/3)(14/3)(5/3) = 2450/27 \approx 90.74.$$

The values of the auxiliary variables are $l = 35/3$, $w = 14/3$ and $d = 5/3$, and these are the size in inches of the length, width and depth, respectively of the largest box.

Exercise 9 Here is the statement: "Postal regulations specify that a parcel sent by parcel post may have a combined length and girth of no more than 108 in. Find the dimensions of a rectangular package that has a square cross section and the largest volume that may be sent through the mail."

Variables are:

l: length in inches
w: width in inches
d: depth in inches
g: girth in inches
V: volume in cubic inches

The colors are meant to suggest that l will be taken as the independent variable, V is the objective, and all other variables are for convenience only. The definition of **girth** gives

$$g = 2d + 2w,$$

and the information that the cross-section is a square gives

$$d = w.$$

Combining these, $g = 4d = 4w$, or

$$d = w = \frac{g}{4}.$$

The objective function $V = lwd$ from general principles concerning volume. The restriction that $l + g \leq 108$ needs to be replaced by an equation. If w and d are **fixed** and l is **increased** from any allowed value to

$108 - g$, then V also increases, so the maximum volume cannot have $l + g < 108$. Thus, we may take $l + g = 108$, or.

$$g = 108 - l.$$

Finally, $V = lwd$, and all factors can be expressed in terms of l to give

$$V = l\left(\frac{108 - l}{4}\right)^2 = \frac{1}{16}(11664l - 216l^2 + l^3)$$

The feasible values of l are $0 \le l \le 108$, and $V = 0$ at both endpoints of this interval, while it is positive inside the interval. This requires that the maximum be taken at an interior critical point. To find the critical points, differentiate to obtain

$$\frac{dV}{dl} = \frac{1}{16}(11664 - 432l + 3l^2) = \frac{3}{16}(3888 - 144l + l^2) = \frac{3}{16}(36 - l)(108 - l)$$

The only **interior** critical point is $l = 36$. The values of the other quantities are $g = 72$ and $w = d = 18$, giving $V = 11664$. This is the volume of the largest such mailable package in cubic inches.

Remark: It would have been easier to factor dV/dl if we had used the product rule to differentiate the factored form of V. The steps of that are

$$\begin{aligned}\frac{dV}{dl} &= l \cdot 2\left(\frac{108 - l}{4}\right)\left(\frac{-1}{4}\right) + \left(\frac{108 - l}{4}\right)^2 \cdot 1 \\ &= \frac{108 - l}{4}\left(-\frac{l}{2} + \frac{108 - l}{4}\right) \\ &= \frac{108 - l}{4}\,\frac{108 - 3l}{4}\end{aligned}$$

Exercise 11 This is the same as Exercise 9 except that the cross-section is a circle instead of a square. It seems more natural in this case to use the **radius** of the circular cross-section as the independent variable, and to call it r. Then, the **girth**, $g = 2\pi r$, and we can restrict, as in Exercise 9 to $l = 108 - g = 108 - 2\pi r$. The **objective** to be maximized is the volume,

$$V = \pi r^2 l = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3.$$

The **feasible region** is $0 \le r \le 54/\pi$, and $V = 0$ at the endpoints while being positive inside the interval. The maximum will be attained at an **interior critical point**. To find that point, differentiate to obtain

$$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2,$$

which is zero at the endpoint $r = 0$, and the interior point $r = 36/\pi \approx 11.459$. The latter must give the maximum volume since there are no other critical points. At this point, $g = 72$, so $l = 36$ and $V = 46656/\pi \approx 14851$. This is the volume in cubic inches of the largest mailable cylinder — more than 27% larger than the largest mailable rectangular box.

Remark: One cannot help but notice that the value of l is the same for both problems. Calculus provides an easy explanation of this, provided we are willing to use l as the independent variable in Exercise 11 as well as in Exercise 9. The volume of the cylinder is

$$V = \frac{1}{4\pi} l(108 - l)^2,$$

which is a **constant multiple** of the expression in Exercise 9. Thus, for each length, the ratio of the volume of the cylinder to that of the box is $4/\pi$, and the largest value occurs for the same value of l.

Exercise 10 The statement, "A production editor decided that the pages of a book should 1 in. margins at top and bottom and ½ in. margins on the sides. She further stipulated that each page should have an area of 50 square inches. Find the page dimensions that will result in the largest printed area.", suggests four variables:

- u: width of page in inches
- v: length of page in inches
- x: width of printed area in inches
- y: length of printed area in inches

Here, x was **elected** independent variable by the class. Since there are margins on **all** sides, the dimensions of the page exceed that of the printed area by twice the width of the margin. Thus, $u = x + 1$ and $v = y + 2$. The constraint on area of the page says $uv = 50$. The **objective** to be maximized is the printed area $A = xy$. Expressions for all variables in terms of x are

$$\begin{aligned} u &= x + 1 \\ v &= \frac{50}{u} = \frac{50}{x+1} \\ y &= v - 2 = \frac{50}{x+1} - 2 = \frac{48 - 2x}{x+1} \\ A &= xy = \frac{x(48 - 2x)}{x+1} = \frac{48x - 2x^2}{x+1}. \end{aligned}$$

The **feasible region** is the interval of values of x for which all dimensions are positive, so

$$0 \le x \le 24.$$

Note that A is zero at the endpoints of this interval and positive in the interior, so the maximum will occur at a critical point. Differentiating the expression for A,

$$\begin{aligned} \frac{dA}{dx} &= \frac{(x+1)(48-4x) - (48x - 2x^2)(1)}{(x+1)^2} \\ &= \frac{48 - 4x - 2x^2}{(x+1)^2} \end{aligned}$$

To find where this is zero, **factor the numerator** to get $2(6 + x)(4 - x)$. The only critical point in the feasible region is $x = 4$, and the expressions for the other variables yield $u = 5$, $v = 10$, $y = 8$, $A = 32$.

Remark: The quotient rule can be avoided by using u as independent variable. This leads to

$$A = \frac{(u-1)(50-2u)}{u} = \frac{-50+52u-2u^2}{u}.$$

Dividing the denominator into **each term** give $A = -50u^{-1} + 52 - 2u$, so $dA/du = 50u^{-2} - 2$. This is a slightly simpler way to find that $u = 5$, from which the values of the other quantities maximizing the printed area can be found.

Exercise 16 The problem statement was: "An apple orchard has an average yield of 36 bushels of apples per tree if tree density is 22 trees per acre. For each unit increase in tree density, the yield decreases by 2 bushels. How many trees should be planted to maximize the yield?"

This needs some clarification since the word **yield** is used in two senses in the statement. The yield that decreases by 2 bushels is the **yield per tree**. The first part of the problem statement then gives the constraint

$$\text{yield} - 36 = -2(\text{density} - 22),$$
$$\text{or yield} = 80 - 2 \cdot \text{density}.$$

The use of the word at the very end of the statement defines an **objective** to be maximized that is **total yield per acre**, the product of **yield per tree** with the **density** in **trees per acre**. This objective is $80(\text{density}) - 2(\text{density})^2$. Differentiation with respect to density gives $80 - 4(\text{density})$, so the greatest yield corresponds to a density of 20 trees per acre.

We were not given enough information to know whether we should accept this answer. Since we were only told about the behavior of the yield when density is **increased** from a base of 22 trees per acre, the model may not be valid for **fewer** trees. In this interpretation, every increase in density decreases the total yield, so we should plant the fewest trees allowed, which is 22 trees per acre. We will then need to be content with a total yield of 792 bushels instead of 800 allowed in the extended model.

Exercise 12 This was not included in the prepared slides, but there was some time available to discuss another problem and one was improvised that resembled this one. Here is the statement: "For its beef stew, the Betty Moore Company uses tin containers that have the form of right circular cylinders. Find the radius and height of a container if it has a capacity of 36 in.^{3} and is constructed using the least amount of metal."

This asks for the **least surface area** for a **given volume**. There is a related problem that asks for the **largest volume** for a **given surface area**. If one uses a method based on implicit differentiation (whose general form is known as the method of **Lagrange multipliers**), a single relation can be found that expresses the property that one of these quantities has a critical point when the the other is constrained to be constant. This works because differentiating a constraint gives an equation for the derivative of one variable with respect to the other that is always valid while treating the quantity as an objective and seeking a critical gives **the same** equation in which the derivative is to be obtained from using the other relation as a constraint. These expressions frequently lead to **natural characterizations** of the optimum. It will still be necessary to solve this together with the constraint to obtain a numerical optimum.

For a cylinder, the equations and their derivatives with respect to r, followed by the consequences of setting derivatives of V and A to zero are

$$V = \pi r^2 h$$
$$\frac{dV}{dr} = \pi r^2 \frac{dh}{dr} + 2\pi r h$$

$$A = 2\pi r^2 + 2\pi r h$$
$$\frac{dA}{dr} = 4\pi r + 2\pi h + 2\pi r \frac{dh}{dr}$$
$$\frac{dh}{dr} = -\frac{2h}{r} = -\frac{h+2r}{r}$$

The optimum requires $2h = h + 2r$, which simplifies to $h = 2r$. Substituting this expression for h gives $V = 2\pi r^3$ and $A = 6\pi r^2$.

For this exercise, $V = 36$, so $r = (18/\pi)^{1/3} \approx 1.7894$, $h = 2(18/\pi)^{1/3} \approx 3.5788$, and $A = 6\pi(18/\pi)^{2/3} \approx 60.3554$.

Remark: It is not clear why this problem was not given the **calculator icon** that was awarded to several easier problems. Here is a direct solution of the given exercise. Solving $V = 36$ for h gives $h = (36/\pi)r^{-2}$, so that $A = 2\pi r^2 + 72/r$. The critical point is given by

$$0 = \frac{dA}{dr} = 4\pi r - 72r^{-2}$$

which gives $r = (18/\pi)^{1/3}$ as before.