Section 5.6 A fundamental property of the exponential function it that the change in the value of the function over a fixed interval is given by a **multiplier** that **depends only on the length of the interval** and not on its starting value. In most models, the **independent variable** is **time**, so that multiplier depends on the interval of time and not how the clock was set. The significant feature of the model is that the change be given by a **multiplier** and not some other type of growth (or decay).

The derivation of this feature of the exponential function is simple. Start from $P(t) = P_0 e^{ct}$, where P_0 is a convenient name for one of the parameters in the model since $P(0) = P_0$. The other parameter c, called the **growth constant** in the text, defines the relation between the **unit of time** used by the clock and the **multiplier** in the process being modeled. This is the form that is easiest to use in calculus — other forms appear when one wants to give an elementary description between the process and familiar units of time. Here is the proof of this property.

$$\begin{aligned} P(t+h) &= P_0 e^{c(t+h)} \\ &= P_0 e^{ct+ch} \\ &= P_0 e^{ct} \cdot e^{ch} \\ &= P(t) \cdot e^{ch} \end{aligned}$$

In particular, the multiplier is identified as e^{ch}, so c can be obtained by dividing the logarithm of the multiplier by h.

Exercise 1 Here

$$Q(t) = 400e^{0.05t},$$

where t represents **time in minutes** (from an undisclosed starting time). The first part of the exercise asks to identify the **growth constant** 0.05 and **initial quantity** 400 by **looking at the formula**. Then, you are to use a calculator to tabulate the function at given values. Here are the results (rounded to a convenient accuracy).

$$\begin{aligned} Q(0) &= 400 \\ Q(10) &= 659.5 \\ Q(20) &= 1087.3 \\ Q(100) &= 59,365.3 \\ Q(1000) &= 2.07 \times 10^{24} \end{aligned}$$

This illustrates that **values of the exponential function easily get beyond the ability to count**. Your calculator uses **scientific notation** to represent numbers whose decimal point does not fit on the same display with the most significant part of the number. We write this with a factor of a power of 10 to tell us how far to shift the decimal place from where it is shown; the calculator separates this exponent from the main part of the number by an "**E**", which should catch your eye in the middle of a decimal number. How would we **say** the value of $Q(1000)$? The names of large numbers in American English (British usage is different) uses "million" for 10^6, and then counts in Latin for each new factor of 10^3: 10^9 is a "billion", 10^{12} a "trillion", etc. Since we four more steps in this sequence to reach 10^{24}, the name of that number would be derived from the Latin name for 7. Thus, we can say that $Q(1000)$ is a little more than **two septillion**.

Exercise 5a We are told that the World Population (of humans) is growing at 2% per year. One interpretation of this is that when time is measured in years, the **growth constant** in the exponential model. That is, $P(t) = P_0 e^{0.02t}$. This interpretation is justified by the fact that this function has the property

$$P'(t) = 0.02 P_0 e^{0.02t} = 0.02 P(t).$$

In other words, the **instantaneous** rate of growth per year is 2% of the current population. A competing interpretation would be that in one year, the population grows by 2%, so that is **multiplied by** 1.02. This would give a growth constant of $\ln 1.02 \approx 0.0198$. The difference is small, so either can be used in informal work. (If money were involved, the definition would need to be more precise, so banks will use a number c to calculate **continuous compound interest** while reporting a larger rate of $e^c - 1$ as an **effective annual yield**.)

This part of the problem asks for the time required for population to **triple** at this rate. Note that only an **interval** of time is mentioned, since the starting time is irrelevant. This requires only finding h so that

$$e^{0.02h} = 3.$$

Since $0.02 = 1/50$, dividing by 0.02 is the same as multiplying by 50, giving $h = 50 \ln 3 \approx 55$.

Part b asks for the effect of reducing the growth constant to 1.8%. A similar computation gives $h \approx 61$, and a look behind the calculation shows that multiplying the growth constant by 0.9 will cause the time to achieve a particular result to be divided by 0.9.

Exercise 11 This exercise encourages you to see how easy it is to calculate with an idea whose formulation won a Nobel Prize. The idea was that while Carbon-14 in isolation is radioactive and decays with a half-life of 5770 years, the ratio of the amount of Carbon-14 to the amount of normal Carbon in a living being exchanging Carbon with the larger environment should be an **equilibrium ratio** that has been essentially constant for 50000 years. This allows us to determine how much Carbon-14 was in a fossil at the time it died and became isolated from the larger environment.

In this exercise, we are told that a sample of wood has 20% of what has been determined to be the original amount, and asked to determine how long the tree that was the source of the wood has been dead. First, we find that time x in units of the C-14 half life. The definition of half-life, and the conversion $20\% = 1/5$ gives the equation

$$\begin{aligned} \left(\frac{1}{2}\right)^x &= \left(\frac{1}{5}\right) \\ \log\left(\frac{1}{2}\right) \cdot x &= \log\left(\frac{1}{5}\right) \\ -\log 2 \cdot x &= -\log 5 \\ x &= \frac{\log 5}{\log 2} \approx 2.3219 \end{aligned}$$

Any base of logarithms may be used in this calculation since a ratio of logarithms is independent of the base. (On my calculator, the log key gives $\log_{10}$ and the ln key gives natural logarithm. The results from different choices of logarithm differ by in the last decimal place, which happened to be the eleventh decimal place, but this is a symptom of work with functions that can only be approximated and numbers that needed to be rounded off between uses. Normally calculators keep extra accuracy that is not shown, so it is rare to see even this much difference in a simple calculation.)

Now, the text book says that each **C-14 half-life** is 5770 years, so our answer converts to

$$2.3219 \times 5770 = 13397.5$$

years.

Note that using the figure of 0.00012 for the **decay constant** of C-14 leads to a time of 13412 years. This shows the effect of preserving only **two significant figures** in this result (actually, the value is correct to **three** significant figures, and this could be indicated by writing it as 0.000120 or 1.20×10^{-4}).

Exercise 13 Since this is an assigned homework problem, only a brief comment will be made here. The given function

$$Q(t) = 120(1 - e^{-0.05t}) + 60$$

could also be written, by collecting the terms differently,

$$Q(t) = 180 - 120e^{-0.05t}.$$

This shows that $180 - Q(t)$ is an example of **exponential decay**, so that using this to model a **learning curve** is making the optimistic claim that **ignorance decays exponentially**.

The model is only claimed to be valid for $0 \le t \le 20$, and it was written in a way that reflects measurements that would have been made over that time interval. To discover more, you should answer the specific questions that are part of this exercise.

Exercise 17 Here we have an example of the **logistic model** used to describe the spread of an epidemic. The formula produced by the model is

$$Q(t) = \frac{1000}{1 + 199e^{-0.8t}}$$

with t being **time in days into the epidemic**. Some values of this function are

$$Q(0) = \frac{1000}{1 + 199} = 5$$
$$Q(1) = \frac{1000}{1 + 199e^{-0.8}} \approx 11$$
$$Q(10) = \frac{1000}{1 + 199e^{-8}} \approx 937$$
$$\lim_{t\to\infty} Q(t) = \frac{1000}{1} = 1000$$

Here, the interpretation of these values is: $Q(0)$ represents the number of cases **just before** the outbreak was noticed; $Q(1)$ is the number of cases at the end of the first day as requested in **(a)**; $Q(10)$ is the number of cases after ten days as requested in **(b)**; the limit approximates the value after a **large number of days**. The words use to describe the model may not be precise enough to lead everyone to choose the same values of t in attempting to answer the questions, but once there is agreement on the value of t, there should be agreement on the value of $Q(t)$. Notice also that $t = 10$ is already large enough to give a value that resembles the limit.

Logistic generalities The **logistic model** wasn't just **dreamed up**. It was derived as the solution of a **differential equation**. While that derivation is beyond the scope of this course, **discovering the equation** that was the basis of the model is a useful illustration of the calculus we have already developed. That is, we can compute the derivative of a general logistic function and rewrite it in a form that reveals the property that makes it suitable as a model.

Start with

$$y = \frac{A}{1 + Be^{-kt}}$$

with A, B and k all positive. Then, as $t \to -\infty$, the exponential term becomes very large, causing the whole denominator to become large and $y \to 0$. Also, as $t \to +\infty$, the exponential term approaches 0, so $y \to A$. Each model will have a fixed value of A that represents an eventual value of y that is part of the model (even if it must be discovered by observation of the process being modeled). The value of B only serves to set the clock, i.e., it interprets the time $t = 0$. If one were to write $B = e^{kt_0}$, then y would be $A/(1 + e^{k(t-t_0)})$. Now, differentiate.

$$\begin{aligned}\frac{dy}{dt} &= A(-1)\left(1 + Be^{-kt}\right)^{-2} B(-k)e^{-kt} \\ &= \frac{kABe^{-kt}}{\left(1 + Be^{-kt}\right)^{-2}} \\ &= k \cdot \frac{A}{1 + Be^{-kt}} \cdot \frac{Be^{-kt}}{1 + Be^{-kt}}\end{aligned}$$

In the last line, one factor of y has been identified. The remaining factor can also be expressed in terms of y. A little algebra reveals that it is $(A - y)/y$. Thus,

$$\frac{dy}{dt} = \frac{ky(A - y)}{A}$$

This shows that the change in y for this function is proportional to both y and $A - y$, and that the change in y resembles ky when y is small and it resembles $h(A - y)$ when y is close to A.