Section 6.2 **Exercise 2** Find

$$\int 4x(2x^2+1)^7\,dx.$$

Before solving this by **finding a useful substitution**, note that there is a **straightforward** (if slightly **painful**) solution that begins by expanding $4x(2x^2+1)^7$ to a polynomial of degree 15 (the polynomial is $512x^{1}5+1792x^{1}3+2688x^{1}1+2240x^9+1120x^7+336x^5+56x^3+4x$) and integrating term-by-term to get a polynomial of degree 16 (which is $32x^{1}6+128x^{1}4+224x^{1}2+224x^{1}0+140x^8+56x^6+14x^4+2x^2$. Whatever method we use to find this integral, we must get something that differs from this only by a constant. In particular, the fact that the degree of the polynomial is 16 and a few other properties of the polynomial are easily seen without **explicit** expansion of the given integrand.

The method that you are **supposed to use** because you found the exercise in this section of the textbook is **substitution**. That is, we introduce

$$u = 2x^2+1$$

which leads by differentiation to

$$du = 4x\,dx.$$

These factors are found (**exactly**) in the original expression for the integral, so we have

$$\int u^7\,du = \frac{1}{8}u^8 + C$$
$$\int 4x(2x^2+1)^7\,dx = \frac{1}{8}(2x^2+1)^8 + C.$$

Here, the last step consists of **composing** the expression of the integral in terms of u that we have found with the definition of u in terms of x that we recognized as the key to solving the problem.

Although we wrote $+C$ at the end of both solutions that we found, these are not the same C. The term-by-term method led to a polynomial without constant term, but $(2x^2+1)^8/8$ has constant term $1/8$. All **other** terms in the two polynomials are the same.

Exercise 6 Find

$$\int \frac{3x^2+2}{(x^3+2x)^2}\,dx$$

The substitution is

$$u = x^3+2x$$
$$du = (3x^2+2)\,dx.$$

Again, these expressions are found in the given integral, so we have

$$\int \frac{du}{u^2} = -\frac{1}{u} + C$$
$$\int \frac{3x^2+2}{(x^3+2x)^2}\,dx = -\frac{1}{x^3+2x} + C$$

In finding this integral, it is helpful to write $1/u^2$ as u^{-2} to emphasize that the **power rule** is used to find the integral.

Exercise 30 Find

$$\int \frac{e^{-1/x}}{x^2}\,dx.$$

One possible substitution is

$$u = \frac{1}{x}$$
$$du = -\frac{1}{x^2}\,dx.$$

There is a natural factor of $-du$ in the given integral, and identifying the substitution gives

$$\int -e^{-u}\,du = e^{-u} + C$$
$$\int \frac{e^{-1/x}}{x^2}\,dx = e^{-1/x} + C$$

Exercise 36 Find

$$\int \frac{(\ln u)^3}{u}\,du.$$

Using the substitution

$$v = \ln u$$
$$dv = \frac{1}{u}\,du,$$

gives

$$\int v^3\,dv = \frac{v^4}{4} + C$$
$$\int \frac{(\ln u)^3}{u}\,du = \frac{(\ln u)^4}{4} + C$$

Exercise 44 Find

$$\int \frac{e^{-u} - 1}{e^{-u} + u}\,du.$$

There was a hint to use the substitution

$$v = e^{-u} + u$$
$$dv = (-e^{-u} + 1)\,du.$$

This gives

$$\int \frac{-dv}{v} = -\ln|v| + C$$
$$\int \frac{e^{-u}-1}{e^{-u}+u}\,du = -\ln\left|e^{-u}+u\right| + C$$

Exercise 47 Find

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx.$$

In this case, one can substitute something **for** x instead of searching for a pattern. If

$$x = y^2$$
$$dx = 2y\,dy$$

where we want $y \geq 0$ so that $y = \sqrt{x}$. Making this substitution gives

$$\int \frac{1-y}{1+y}2y\,dy = \int \frac{2y-2y^2}{1+y}\,dy.$$

Long division allows the integrand to be rewritten:

$$\frac{2y-2y^2}{1+y} = -2y + 4 - \frac{4}{1+y}.$$

Integrating this gives

$$-y^2 + 4y - 4\ln|y+1| + C = -x + 4\sqrt{x} - 4\ln\left|\sqrt{x}+1\right| + C$$