Section 6.4

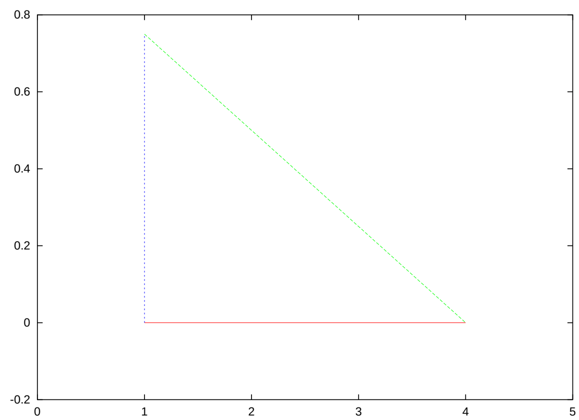
Exercise 4 This exercise asks to find the area **under** $y = -(1/4)x + 1$ and **over** $[1, 4]$ by both **geometry** and **calculus**. Using **calculus** means to evaluate

$$\int_1^4 \frac{1}{4}x + 1\,dx$$

by the **fundamental theorem**. The steps are

$$\begin{aligned}\int_1^4 -\frac{1}{4}x + 1\,dx &= -\frac{1}{8}x^2 + x\Big|_1^4 \\ &= \left(-\frac{16}{8} + 4\right) - \left(-\frac{1}{8} + 1\right) \\ &= \frac{16}{8} - \frac{7}{8} = \frac{9}{8}\end{aligned}$$

A graph is



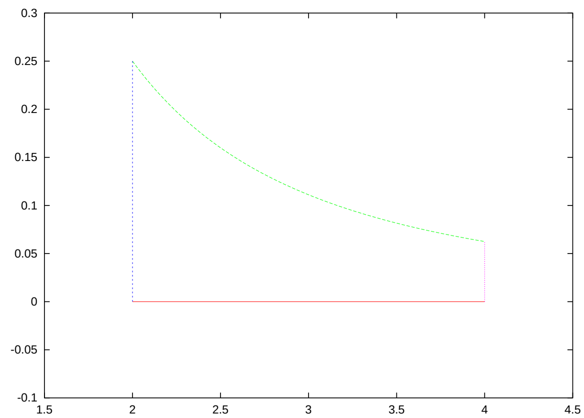
Using **geometry** means to identify $y = -(1/4)x + 1$ as a **line**. Evaluating at the endpoints of the given interval shows that it connects the points $(1, 3/4)$ and $(4, 0)$. The figure is a **triangle** whose **base** has length $4 - 1 = 3$ and whose **height** is $3/4$, which gives

$$\text{Area} = \frac{1}{2} \cdot 3 \cdot \frac{3}{4} = \frac{9}{8}.$$

Exercise 10 We must find the area **under** $y = 1/x^2$ and **over** $[2, 4]$ using calculus. Here are the steps:

$$\begin{aligned} \int_2^4 \frac{1}{x^2}\,dx &= \left.\frac{-1}{x}\right|_2^4 \\ &= \left(\frac{-1}{4}\right) - \left(\frac{-1}{2}\right) = \frac{1}{4} \end{aligned}$$

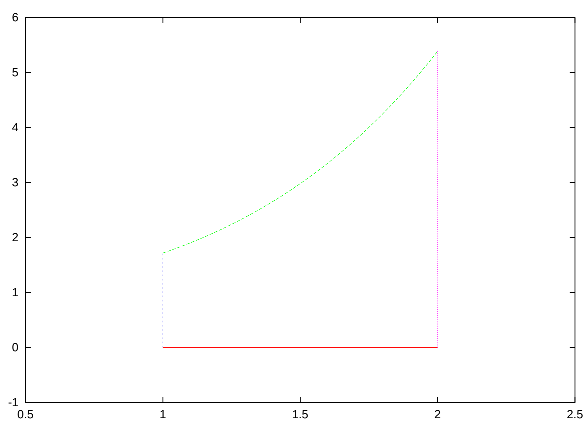
A graph is



Exercise 16 We must find the area **under** $y = e^x - x$ and **over** $[2, 4]$ using calculus. Here are the steps:

$$\begin{aligned} \int_1^2 e^x - x\,dx &= \left. e^x - \frac{1}{2}x^2 \right|_1^2 \\ &= \left(e^2 - \frac{4}{2}\right) - \left(e^1 - \frac{1}{2}\right) \\ &= e^2 - e - \frac{3}{2} \approx 3.17 \end{aligned}$$

A graph is



Exercise 18 Although **area** is often used to **motivate the definition** of the integral by Riemann sums and to **illustrate the proof** of the fundamental theorem for functions with a positive range, the definition and proof are valid for **all real valued functions**. A simple example is

$$\int_{-1}^{2} -2\,dx = -2x|_{-1}^{2} = (-4) - (2) = -6$$

A picture of this region would show a rectangle of width 3 and height 2 **below** the x axis. This explains the **absolute value** of the answer being 6. A theory of **signed area** involves declaring the **left to right** direction on the x axis and the **upward** direction on the y axis as **positive**, and **changing sign with each reversal of direction**. Since this rectangle is behind us as we look up the y axis, ahile the x axis is traced in the positive direction, the signed area is negative.

The most important property of signed area is that it is a **consequence of tne definition** of the integral, not an arbitrary invention.

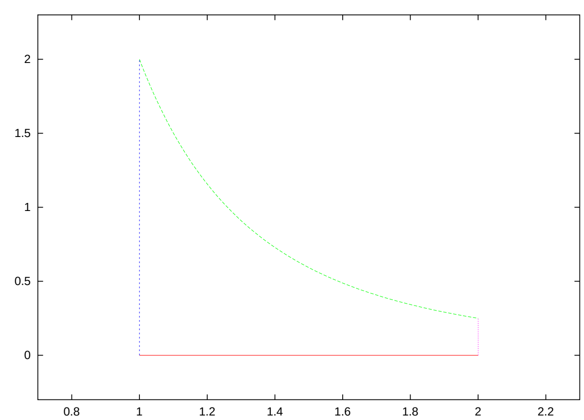
Exercise 36 Another evaluation.

$$\int_{1}^{2} \frac{2}{x^3}\,dx = \left.\frac{-1}{x^2}\right|_{1}^{2} = \left(\frac{-1}{4}\right) - \left(-1\right) = \frac{3}{4}$$

Although the functions appearing in this summary are written as fractions with positive powers of x in the denominator, the calculus should be **done** by rewriting them using negative exponents. The relevant differentiation is

$$\frac{d}{dx}\left(x^{-2}\right) = -2x^{-3},$$

which needs to have the expressions on **both sides** multiplied by -1 in order to identify the given integrand as a derivative. A graph is



Exercise 45 Here, we are given that the velocity of a boat seeking a water speed record was given by

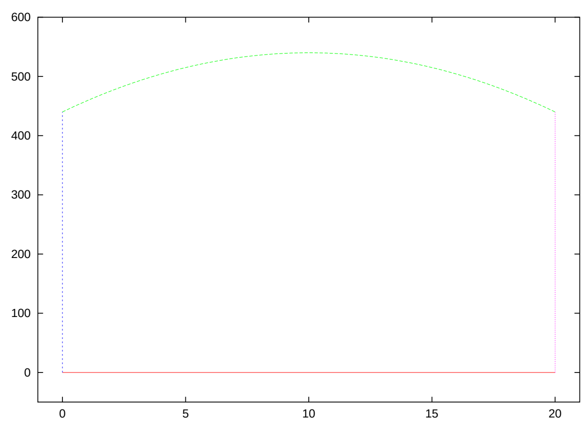
$$v(t) = -t^2 + 20t + 440 \quad (0 \le t \le 20)$$

where t is the **time in seconds after activating a booster rocket** and v is the **velocity in feet per second**. The choice of scales allow **all time to be measured in seconds** and **all distances to be measured in feet**. The time $t = 0$ is identified by the **event of activating a booster rocket**. The exercise asks us to use this information to determine the **distance traveled during the 20 seconds following the activation of the booster rocket**. The interpretation of this as an integral is given as a hint, but this is easily derived. If x denotes the **distance in feet from the position when the booster rocket was activated**, then $dx/dt = v$ and $x(0) = 0$. The fundamental theorem says that $x(t)$ is an integral of $v(t)$ which becomes **uniquely defined** by specifying $x(0)$. The distance traveled is

$$\int_0^{20} -t^2 + 20t + 440\, dt = -\frac{t^3}{3} + 10t^2 + 440t\Big|_0^{20} = \left(-\frac{20^3}{3} + 10 \cdot 20^2 + 440 \cdot 20\right) - (0) = 10133.3\,.$$

This is the **distance in feet**. The **international nautical mile** (equal to 6076.11549 feet) is a more appropriate unit for distances of this nature. Conversion gives a distance of a little more than $1\,^2\!/_3$ *nmi*.

A graph is



This gives a **real application of calculus**! If you plan to set a new speed record, you need lots of space. The distance calculated here is only part of the story, since the boat had already attained a speed of 440 feet per second before activating the booster rocket.

Section 6.5 This section finds definite integrals in cases where some work is needed to find the corresponding indefinite integral. In particular, when a substitution is required, an alternate method is given that does not require an expression for the indefinite integral. Although this is useful when many such integrals need to be found, it can be confusing with the little time available for the topic in this course. Only the solution using the complete determination of an indefinite integral will be used here.

Exercise 8 To find

$$\int_0^2 \frac{x}{\sqrt{x^2+5}}\,dx,$$

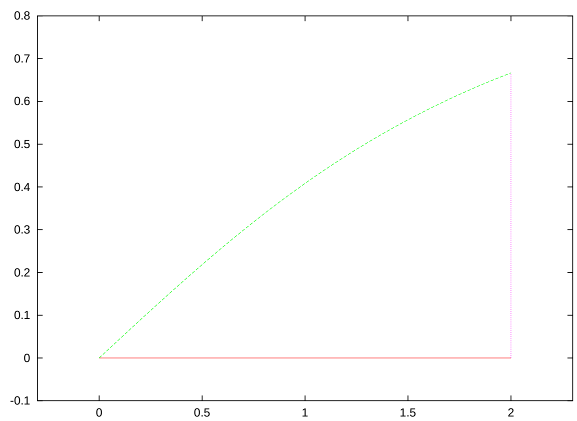
one should substitute $u = x^2+5$. This requires $du = 2x\,dx$, so that $x\,dx = (1/2)\,du$. This relates indefinite integrals leading to an antiderivative:

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+5}}\,dx &= \int \frac{1}{2}u^{-1/2}\,du \\ &= u^{1/2} \\ &= \sqrt{x^2+5}.\end{aligned}$$

Since the goal is a **definite integral**, only one antiderivative is needed, so an arbitrary constant has not been added. Now,

$$\sqrt{x^2+5}\,\Big|_0^2 = \sqrt{9}-\sqrt{5} = 3-\sqrt{5} \approx 0.764.$$

A graph is



(When done in lecture, the limits of integration were incorrectly transcribed as 0 and 8, leading to an integral of $\sqrt{69}-\sqrt{5}$.)

Exercise 11 To find

$$\int_{-1}^{1} x^2(x^3+1)^4\,dx$$

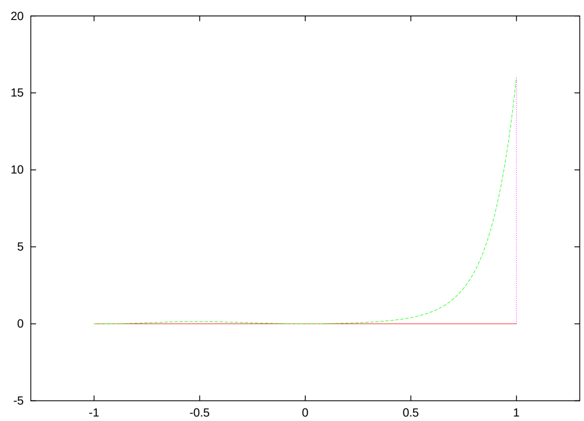
one should substitute $u = x^3 + 1$. This requires $du = 3x^2\,dx$, so that $x^2\,dx = (1/3)\,du$. This relates indefinite integrals leading to an antiderivative:

$$\begin{aligned} \int x^2(x^3+1)^4\,dx &= \int \frac{1}{3}u^4\,du \\ &= \frac{u^5}{15} \\ &= \frac{(x^3+1)^5}{15}. \end{aligned}$$

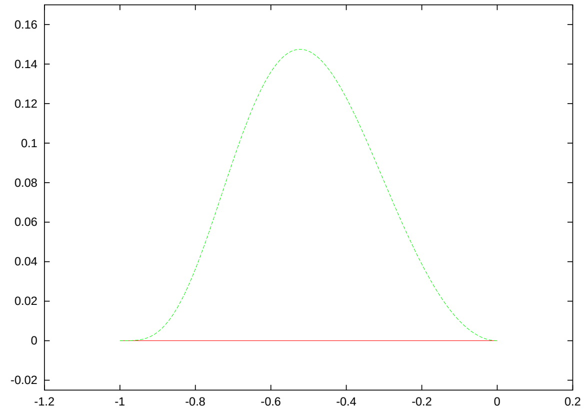
Now,

$$\left.\frac{(x^3+1)^5}{15}\right|_{-1}^{1} = \frac{2^5}{15} - \frac{0}{15} = \frac{32}{15} \approx 2.13.$$

A graph is



Although this accurately shows that the contribution of negative values of x is barely noticeable (computing that integral shows that it contributes $1/15$ which is $1/32$ of the total integral), this part of the graph is the most interesting from the point of view of **graph sketching**. The algebra needed to identify the critical points and inflection points of this graph is best done with the help of a computer algebra system, but the calculus is straightforward. Here is the graph



Exercise 28 To find

$$\int_1^2 \frac{\ln x}{x}\,dx$$

one should substitute $u = \ln x$. This requires $du = (1/x)\,dx$, giving

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int u\,du \\ &= \frac{u^2}{2} \\ &= \frac{(\ln x)^2}{2}\end{aligned}$$

Now,

$$\left.\frac{(\ln x)^2}{2}\right|_1^2 = \frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{(\ln 2)^2}{2} \approx 0.24.$$

Here is the graph