Section 2.5 Continuity Calculus books pretend that continuity is defined in terms of limits, and that limits are things that can actually be found by some sort of **process**. The introduction of limits before considering continuity reinforces that claim. One reason for doing this is that the definition of the derivative **requires** a process that fills in a missing value.

However, limits are **difficult to understand**, and they would be even **more difficult** if a precise definition was given.

By contrast, continuity is a **natural** idea. When working with functions, mathematicians frequently need to require that a function be **nice** before they can conclude that it will be **useful** for some application. The definition of function is made very general to separate those features that only depend on the ability to compose functions from deeper properties that need to be considered when the domain is a set of real numbers.

In a sense, continuity expresses the idea that a function is **computable**. Although calculators now have large memory, it is **finite**. We talk about real numbers as infinite decimal expansions, but only a handful of

digits will be used to represent a number in a calculator. For a calculation to be useful, the operation on the digits we have in our hand must say something about the value we would get if we were applying the operation on the number we claimed to have.

Numbers like π are fascinating because there is no simple description of their decimal representation. A few years ago, the Chudnovsky brothers studied some formulas that improved the ability to produce the decimal expansion of π. As a test, they found **2 billion** digits. The magnitude of this computation exercised muscles their computer never knew it had. They also had some difficulty finding a place to put the result. By contrast, your calculator claims to know π, but it will never give you more than a dozen or so digits of the decimal expansion. This leads to fundamental inaccuracy in some computations.

Continuity requires that a good approximation to the value of a function can be achieved by using a good approximation to the input. A proof of continuity usually includes some information about how good an approximation will be required.

The connection between continuity and limits is that a function f is continuous if, for all a, the limit of $f(x)$ as $x \to a$ is $f(a)$.

What you need to know about continuity is that constant functions and the identity function $f(x) = x$ are continuous, and that algebraic operations like addition and multiplication preserve continuity. So does division, provide the denominator stays away from zero. From this it follows that **all polynomials** are continuous, and **all rational functions**, away from the points that make the denominator zero, are continuous.

We are biased in favor of nice functions. Since we expect to compute values of functions that we use, the functions with names tend to be continuous. The simplest example of a function that is **not** continuous is

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \text{ is positive} \\ 0 & \text{if } x \text{ is zero} \\ -1 & \text{if } x \text{ is negative} \end{cases},$$

and this function is discontinuous only at zero. That is, if you are told that a number x is **close to** a positive number like 1, you expect that x is positive, but if you

are told that x is close to zero, you have no information about its sign.

One-sided limits Although there are other ways in which a function may fail to be continuous, the **jump discontinuity** of $\operatorname{sgn}(x)$ is the most common example. To describe such a jump at $x = a$, it is common to say that there **would be** a limit of $f(x)$ if we approached a only through smaller values, and to introduce the notation

$$\lim_{x \to a^-} f(x)$$

to have a name for the result. Similarly,

$$\lim_{x \to a^+} f(x)$$

refers to the limit of $f(x)$ when a is approached only through larger values. Note that

$$\lim_{x \to 0^-} \operatorname{sgn}(x) = -1 \qquad \lim_{x \to 0^+} \operatorname{sgn}(x) = +1.$$

with $\operatorname{sgn}(0) = 0$ **A notorious example** Normally, when a function is defined by cases, there is an attempt to make it continuous. The **Income Tax Rate**

Schedule for Married Taxpayers in the State of New Jersey is an exception. One suspects that the intend was to make it continuous, but the following description has been published for several years (even after being informed).

The computation is in the form of a worksheet, but it is easy to translate it into an algebraic description:

$$NJ(x) = \begin{cases} 14x & \text{if } 0 < x \leq 20 \\ 17.5x - 70 & \text{if } 20 < x \leq 50 \\ 24.5x - 420 & \text{if } 50 < x \leq 70 \\ 35x - 1154.50 & \text{if } 70 < x \leq 80 \\ 55.25x - 2775 & \text{if } 80 < x \leq 150 \\ 63.7x - 4042.50 & \text{if } 150 < x \end{cases} .$$

where x is taxable income in thousands of dollars. **Applications**. A continuous function has no jumps. The **intermediate value theorem** give a precise statement: If f is continuous on $[a, b]$ and M is a number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ such that $f(c) = M$.

The general statement can be reduced to the special case in which $M = 0$, $f(a) < 0$ and $f(b) > 0$.

The formal definition of continuity allows a proof which includes a method for finding c called the **bisection method**. The idea of the proof is to find $Q = f\big((a+b)/2\big)$. Then, if $Q = 0$, $c = (a+b)/2$; if $Q < 0$, look for c in the smaller interval (half the size of the original) $[(a+b)/2, b]$; and if $Q > 0$, look in $[a, (a+b)/2]$. Repeating gives an arbitrarily small interval on which the function changes sign. These intervals have an intersection that is a single point (we have essentially described the binary expansion of a point in the intersection), and the value of f there is arbitrarily close to both positive and negative numbers. This only happens if the value is zero.

For this method to locate a point where $f(x) = 0$, it is only necessary to be able to calculate $f(x)$ for given x. If you start from an interval of length 1, you get c to 12 decimal place accuracy in about 40 steps. The description of rootfinding methods on calculators indicates that this is the method used.