Section 3.1 Basic Differentiation formulas. The derivative **at a point** is defined as the limit of the **difference quotient** determined by that point and a nearby point. We expect that it will **often** be possible to perform this construction at all points of the domain of the given function. This gives a new function defined on **the same** domain. If the original function is called $f(x)$, the derivative will be denoted $f'(x)$ If we want a computation that ends up with $f'(x)$ instead of $f'(a)$ (as in most of the examples done in the last lecture), then the nearby point should be $\big(x+h,\ f(x+h)\big)$. With this notation, it is important to remember to specialize x to a given value after finding the derivative when working with an application that requires the value of the derivative at a particular point.

The derivative of a constant function. Suppose $f(x) = c$. Then

$$f(x+h) = c$$
$$f(x+h) - f(x) = c - c = 0$$
$$\frac{f(x+h) - f(x)}{h} = 0$$

and the limit is zero. Thus,

$$f'(x) = 0.$$

The derivative of the identity function. Here,

$$f(x) = x.$$

Then

$$f(x+h) = x + h$$
$$f(x+h) - f(x) = (x+h) - (x) = h$$
$$\frac{f(x+h) - f(x)}{h} = h/h = 1$$

and the limit is 1.

The derivative of the squaring function. Now take

$$f(x) = x^2.$$

Then

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$
$$f(x+h) - f(x) = 2xh + h^2$$
$$\frac{f(x+h) - f(x)}{h} = 2x + h$$

and the limit is $f'(x) = 2x$.

The general power rule. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

In this generality, the proof is difficult. If n is a positive integer, the result can be proved by expanding $f(x+h)$ by the binomial theorem. In particular, $f(x+h)$ will contain terms x^n and $nx^{n-1}h$, and all other terms will have a higher power of h. The first term drops out, when $f(x+h) - f(x)$ is formed, and the second term gives the expression that we expect to be the limit. When any other term is divided by h, there will still be a positive power of h in the difference quotient. Such terms go to zero in the limit.

The linearity property of the derivative. We want the derivative of $r(x) = cf(x)$ or $s(x) = f(x)+g(x)$, where the derivatives of $f(x)$ and $g(x)$ are assumed to

be known. Here are the calculations of the derivatives:

$$\begin{aligned}
r(x+h) &= cf(x+h) \\
r(x+h) - r(x) &= c\big(f(x+h) - f(x)\big) \\
\frac{f(x+h) - f(x)}{h} &= c\frac{\big(f(x+h) - f(x)\big)}{h} \\
r'(x) &= cf'(x) \\
s(x+h) &= f(x+h) + g(x+h) \\
\frac{s(x+h) - s(x)}{h} &= \frac{f(x+h) - f(x)}{h} \\
&\quad + \frac{g(x+h) - g(x)}{h} + \\
s'(x) &= f'(x) + g'(x)
\end{aligned}$$

The derivative of a polynomial. Write the polynomial $m(x)$ as a **sum of monomials**, where a monomial is a constant multiple of a power of x. The linearity properties show that $m'(x)$ is the same combination of the derivatives of x. This allows an expression for $m'(x)$ to be found as quickly as writing $m(x)$.

If a polynomial has not been fully expanded, the expansion should be done **first**.

The product rule. Let $p(x) = f(x)g(x)$. Then

$$\begin{aligned} p(x+h) &= f(x+h)g(x+h) \\ p(x+h) - p(x) &= f(x+h)g(x+h) \\ &\quad - f(x)g(x) \\ &= f(x+h)\big(g(x+h) - g(x)\big) \\ &\quad + \big(f(x+h) - f(x)\big)g(x) \\ \frac{f(x+h) - f(x)}{h} &= f(x+h)\frac{g(x+h) - g(x)}{h} \\ &\quad + \frac{f(x+h) - f(x)}{h}g(x) \end{aligned}$$

If $f(x)$ is continuous and both $f'(x)$ and $g'(x)$ exist, then

$$p'(x) = f(x)g'(x) + f'(x)g(x).$$

The thing to notice is that each term has the form of something related to f and something related to g, and there is exactly one derivative in each term.

This may be used to get an expression for the derivative of the **unexpanded** form of a polynomial. When the result is expanded, we know that we should get an equivalent expression to the result of term-by-term differentiation on the expanded form of $p(x)$

What's next? There is also a **quotient rule** that is a little more complicated that the product rule. We will derive this rule from the product rule and the **chain rule** that gives an expression for the derivative of a composition of functions. With these rules, the determination of the derivative of a function uses a set of step-by-step rules for evaluating the function to produce some expression for the derivative. The resulting expression is the simplified.

Since these rules are proved using the definition of the derivative, it follows that two expressions that define the same function will have derivatives that also describe the same function.