Section 3.3 The chain rule. The **chain rule** tells us that if the derivatives of functions f and g are known, then the **composition** $f \circ g$ has a derivative, and there is a **formula** that gives it in terms of f and g and their derivatives. In this notation, the formula looks awkward, so it is better not to write it until we have seen, and **interpreted** a more natural form of the rule.

The linear case. First, let us see how composition of linear functions behaves. Let $f(x) = ax + b$ and $g(x) = cx + d$, and introduce $H(x) = (f \circ g)(x)$. Then $H(x) = (f \circ g)(x) = f\big(g(x)\big)$, so we must replace the x used in the definition of f with the expression $g(x)$ that contains a variable x that is **playing a different role**. Poor x is overworked, and unnecessarily so, since its role in these formulas is just to **illustrate** how the functions are computed. We could equally well have defined the function f by writing $f(y) = ay + b$. To find $(f \circ g)(x)$ we need only substitute $y = g(x) = cx + d$ for the y in this expression. The result is

$$(f \circ g)(x) = a(cx + d) + b = acx + (ad + b).$$

One may be struck by how **complicated** the **constant**

term is, but what should be noted is:

If the derivatives of f and g are constant, then the derivative of the composition $f \circ g$ is the product of those constants.

This statement is deliberately written in words in order to avoid writing a misleading formula. Mathematical statements have both a **hypothesis** and a **conclusion**, and should be used by first **verifying** the hypothesis to justify using the conclusion. Excessive use of formulas for the conclusions of theorems makes the application of the theorem seem like little more the substitution of one expression for another. This is only a simple finish to the process that built the structure that made this possible.

In **applied** calculus, the structure is the important part, not the formulas with which it is decorated.

General formulation. Working with the composition of functions is simplified by introducing different names for the elements in the spaces connected by the functions. The picture associated with $H = f \circ g$ is

$$H\colon \{x\} \xrightarrow{g} \{y\} \xrightarrow{f} \{z\}.$$

(A similar picture was used in lecture 2, but it emphasized the **spaces** connected by the functions as part of a study of the **domain** and **range** of a function.)

When we want to do **Calculus**, we need to **analyze** the relation between the composite function $f \circ g$ and its parts f and g. What our current picture says is that we have

$$y = g(x)$$
$$z = f(y)$$
$$\text{combine to give}$$
$$z = H(x).$$

To study this at $x = a$, we need to produce $b = g(a)$ to get the corresponding value of the variable y that is used to describe the function f. Then, $c = f(b)$ is the value of the variable z giving the value in the range of $H = f \circ g$ corresponding to $x = a$ in the domain.

The main claim of differential calculus is that the behavior of the graph of the function near a particular point can be approximated by behavior of the tangent line at that point. In this example we have three different functions whose graphs live in three different spaces. The function g is graphed in an (x, y) plane in which we have marked the point (a, b) on the graph; the function f is graphed in a (y, z) plane in which we have marked the point (b, c) on the graph; and The function H is graphed in an (x, z) plane in which we have marked the point (a, c) on the graph. The tangent lines to these curves at the marked points are

$$(y - b) = g'(a)(x - a) \qquad (T_g)$$
$$(z - c) = f'(b)(y - b) \qquad (T_f)$$
$$(z - c) = H'(a)(x - a) \qquad (T_H)$$

For the linear approximation to H to be built from the linear approximations to f and g, we would have $H'(a) = f'(b)g'(a)$. Note that f' is evaluated at a different point that the other functions, but that is necessary because f has a different domain than the other functions. When expressed as functions of x,

$$H'(x) = f'\big(g(x)\big) \cdot g'(x).$$

This is a (correct) statement of the (true) chain rule.

A word about the proof. It can almost be proved by writing the difference quotient

$$\frac{H(x+h) - H(x)}{h}$$

as

$$\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

and taking limits. This breaks down if it is possible for $g(x+h) = g(x)$ for small $h \neq 0$. A correct proof requires a technical device to avoid this. The modified proof is reasonably straightforward, but it undermines the importance given to limits in calculus textbooks, so it is often hidden.

The present textbook attempts to avoid proofs entirely. This seems to be misguided. If a proof is too technical, it can be a distraction, but the idea of a proof can help to clarify the statement of a theorem and provide clues to its application. Thus, the extra terms introduced to express a difference quotient for a product of functions in terms of difference quotients of the factors gives an algebraic statement that helps to explain the unusual appearance of the product rule. Similarly, the factorization of the difference quotient for a composition includes another reminder of the need for evaluating f' at $g(x)$.

A different approach. Not only is the importance of the idea of **limit** overrated, but so is the attempt to express all of calculus in terms of **functions**. It took a long time to give an accurate description of the chain rule expressed in the language of functions since the concentration on real-valued functions of a real variable removes the roles played by the domains and ranges of functions appearing in an expression. The insistence on calling every function f and every variable x separates theory from practice with mathematical objects losing their individuality in theoretical discussions. On the other hand, calculators encourage the use of separate names to distinguish the roles of different quantities in a calculation. Complicated expressions can be computed by naming parts of the expression and formulating the dependence of the answer on those parts rather than attempting to write a single formula that emphasizes **only** the algebraic properties of the dependence.

The subtle concept of function was valuable in building a theoretical foundation of calculus, but it is better to **do** calculus in a universe consisting of a single **independent variable** (often x, sometimes t) with an arbitrary number of **dependent variables**, and expressions giving some variables in terms of other variables that could be combined to give everything in terms of the independent variable. In this approach, if y can be expressed in terms of x, the derivative of that dependence (i.e. of the function expressing y in terms of x) is denoted

$$\frac{dy}{dx}.$$

For the moment, this will be a single new complicated name for a variable, and calculus gives an expression for it in terms of x.

If, also, z can be expressed in terms of y, there will be a single new complicated name for a variable

$$\frac{dz}{dy}$$

that can be expressed in terms of y. These pieces allow z to be expressed in terms of x and lead to the introduction of

$$\frac{dz}{dx}.$$

In this notation, the chain rule says that

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$$

which **looks** obvious, and should be easy to remember. It is not as obvious as it looks, since it is still **the same** chain rule described above. The expressions like dy/dx are not fractions, since this is just a complicated name for a derivative, and things like dx and dy that **appear to be** part of the expression have not (yet) been given an independent meaning. Furthermore, the factor dz/dy is defined so that it is found as an expression in terms of y, but everything else is supposed to be expressed in terms of x. However, we have an expression for y in terms of x that can be used in interpreting this factor.

The description has gone on too long. It is time for some examples.

Multiplicative inverses and the quotient rule. Let $z = 1/y = y^{-1}$ and $y = q(x)$. The general power rule gives

$$\frac{dz}{dy} = (-1)y^{-2} = \frac{-1}{y^2},$$

and $dy/dx = q'(x)$. Expressing everything in terms of x.

$$\frac{dz}{dy} = \frac{-1}{y^2} = \frac{-1}{q(x)^2},$$

and

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{-1}{q(x)^2} \cdot q'(x) = \frac{-q'(x)}{q(x)^2}.$$

Thus, if

$$f(x) = \frac{p(x)}{q(x)} = p(x) \cdot \frac{1}{q(x)},$$

then

$$\begin{aligned} f'(x) &= p(x) \cdot \frac{-q'(x)}{q(x)^2} + p'(x) \cdot \frac{1}{q(x)} \\ &= \frac{q(x)p'(x) - p(x)q'(x)}{q(x)^2}. \end{aligned}$$