The main example. We are now ready to apply calculus to the circle. A complete proof that of the claims we are about to make would be difficult. However, as soon as you know that the claims are **true**, they lead to fairly straightforward algebra and calculus. The main example in this section will be the **unit circle** with center at the origin. It was noted in Section 1.3 that the points (x, y) on this curve are characterized as those that satisfy the equation

$$x^2 + y^2 = 1. \qquad (C)$$

Although most of the calculus we have done so far has dealt with functions given by simple formulas whose domains include **large** intervals, the theoretical foundation requires very little. This gap between theory and practice will continue to be present in the new applications introduced here. It need not cause any confusion: the practice of calculus always deals with **expressions** and the special symbols dy/dx even if the theoretical side involves unfamiliar properties of **functions** and **limits**.

The role of **limits** is express that the derivative at a point depends only on what happens near the point. This means that the **technical requirement** that vertical lines meet graphs of functions in at most one point is not a serious restriction.

If the circle (C) is cut along the x-axis, removing the points $(\pm 1, 0)$, the remaining points lie on an **upper semicircle** with y **strictly positive**, and a **lower semicircle** with y **strictly negative**. Each of these arcs is the graph of a function. Indeed, since (C) is a quadratic equation, the quadratic formula allows these functions to be written using nothing worse than a square root.

Even this is more than we need: for any point (x_0, y_0) on the circle with $y_0 \neq 0$, any reasonable function $f(x)$ defined an open subinterval of $(-1, 1)$ containing x_0 with $f(x_0) = y_0$ whose graph lies in the circle could be differentiated to find the slope of the tangent line to the circle at (x_0, y_0).

Tangents to circles were studied in Geometry, so you should already know enough properties to identify the tangent line to a circle. This allows the claims made by calculus to be checked. Moreover, the familiarity of the example, and the simple equation of the tangent that can be found by geometric methods, introduces a requirement that calculus provide a **simple** derivation of the equation of the tangent to a circle.

Why don't we just solve for one variable in terms of the other? Often we can't do this. Even the solution of the quadratic equation required the **invention** of the square-root function. The hope that other fractional powers would be the key to all algebra was destroyed in the mid Nineteenth Century, although the successful solutions of equations of degree 3 and 4 proved to more complicated than was expected. It turned out to be much more useful to have a **very general** definition of function that would be sure to include everything that we would try to compute.

The circle and its tangents. The starting point for using calculus to find the tangents to the circle (C) is simply to **assume that it can be done**. That is to assume that there a differentiable function $f(x)$ defined on **some** interval I around x_0 such that $f(x_0) = y_0$ and $x^2 + f(x)^2 = 1$ on I. It won't even be necessary

to name the function, since the calculus can be done using only the symbols x and y.

Since y is assumed to be a differentiable function of x,

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx}.$$

However, we are also assuming that this function is such that $x^2 + y^2$ is equal to the**constant function** 1, so **its derivative is zero**. The resulting equation,

$$2x + 2y\frac{dy}{dx} = 0,$$

is easily solved for dy/dx to get

$$\frac{dy}{dx} = -\frac{x}{y}.$$

At a point (x_0, y_0) on the circle (i.e., with $x_0^2+y_0^2 = 1$) where $y_0 \neq 0$ (a temporary restriction to allow x to be taken as the independent variable), the tangent line has slope $-x_0/y_0$, so the equation is

$$\frac{y - y_0}{x - x_0} = -\frac{x_0}{y_0}. \qquad (T)$$

On the way to this equation, we found that the slope of the tangent is the **negative reciprocal** of the slope of the **radius** joining the center $(0, 0)$ to the point (x_0, y_0). Also note that the decision to describe the direction of the line using its slope excluded vertical lines and forced $y_0 \neq 0$. Changing our view to take y as the independent variable would include vertical lines while excluding horizontal ones. An approach that is willing to allow a choice of independent variables would restore the perfect symmetry of the circle and show that the equation

$$x_0 x + y_0 y = 1$$

gives a tangent at any point with $x_0^2 + y_0^2 = 1$.

Another view of the quotient rule. Suppose that $f = p/q$ where p and q are expressions in our independent variable — which we don't need to identify if we use D to denote differentiation with respect to it. We have already derived the quotient rule from the definition so we know that f is differentiable if p and q are. That allows the method of implicit differentiation to find Df by creating an equation relating it to p, q and their derivatives.

To do this, clear denominators to obtain $fq = p$ and differentiate using the product rule to obtain

$$(f)(Dq) + (q)(Df) = Dp.$$

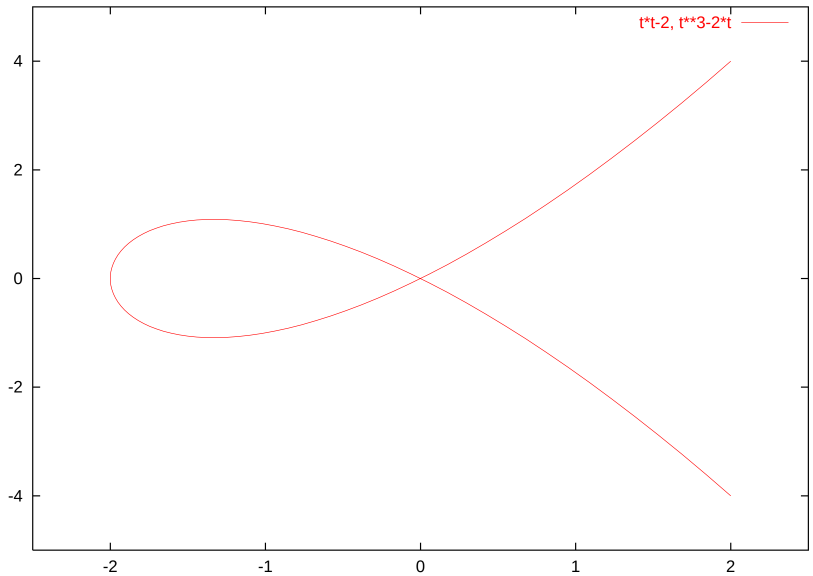
This only unknown in this equation is Df, and solving for it gives

$$Df = \frac{(Dp) - (f)(Dq)}{q} = \frac{(Dp) - (\frac{p}{q})(Dq)}{q}.$$

Recognizing that this requires a denominator of q^2 leads to the familiar formula.

The role of linearity. The last step in an implicit differentiation problem is always to solve an equation for the desired derivative. This equation is obtained by differentiating an **identity** that defines the function we are studying. It is a consequence of the rules for differentiation that this equation will **always** be linear in the quantity we are solving for. The solution will consist of collecting term having this derivative **as a factor** in one place, and everything else in another place, and then dividing.

In the case of a equation involving x and y being used to define y as a function of x, the result will always be an expression for dy/dx in terms of x and y. The **arbitrary decision** to take x as the independent variable means that the process fails at points with vertical tangents. It also fails in cases like $y^2 = x^3 + 2x^2$ at the origin (shown below).



In this example, $2y(dy/dx) = 3x^2 + 4x$, so

$$\frac{dy}{dx} = \frac{3x^2 + 4x}{2y}.$$

The two tangent lines that are **visible** at the origin have slopes given by the values of y/x which is close to $2x/y$ and the value of dy/dx because x^3 is much smaller than x^2 if x is small.

Related rates. If you look closely, you will see that the previous graph was **drawn** by giving the location (x, y) as a function of a new independent variable t. This sort of thing happens frequently: a process causes a point to move in the plane as a function of time, and there is also an equation that tells whether a point (x, y) is **ever** visited by the process. The set of points visited is called the **trajectory** of the process. Differentiating the equation of the trajectory with respect to t gives a relation between x, y, dx/dt, and dy/dt at any point of the trajectory, independent of the actual motion given by the process. If the location and one derivative are given, then the other derivative can be found. There isn't much more to the **theory**

of such problems, so it will be most useful to get directly to some examples. It is worth noting that such problems work with values at a single point. It is not necessary to make any global assumption about the specified derivative. (In some books, the specified derivative is often assumed constant, but this is not necessary. The present text describes these problems at the correct level of generality.)