Concavity. We have seen that a function being increasing (more or less) corresponds to its derivative being positive. Applying this to $f'(x)$ gives that **the derivative of a function is increasing when the second derivative of the function is positive**. The textbook **defines** the phrase **concave upward** as the property of having an increasing derivative, so it has the theorem:

If $f''(x) > 0$ on an interval, then f is concave upward on that interval.

Applying these considerations to $-f$ gives a property called **concave downward** that corresponds to a negative second derivative.

Some pictures are shown to suggest that this is a property that can be **seen** in a graph.

You can easily produce your own pictures by noting that for $f(x) = x^2$, one has $f'(x) = 2x$ and $f''(x) = 2$. The second derivative is everywhere positive in this case.

Note that the change $x \to -x$ **preserves** concavity although it **reverses** the property of being increasing

or decreasing. The notation for second derivatives hints at why this should be the case with the use of a dx^2 factor.

Inflection points. If $f''(x) = 0$ at a point $x = x_0$, we cannot determine the concavity at that point. If $f(x)$ is **linear**, then $f''(x) = 0$ **everywhere**. Otherwise, the places where $f''(x) = 0$ are likely to be **isolated**. If we have some interval around x_0 such that $f''(x) \neq 0$ except at x_0, then we can ask about the sign of the second derivative at other points.

It turns out that derivatives always have the **intermediate value property**, so $f''(x)$ will have the same sign at all points less than x_0 in our interval. Similarly, it will have the same sign at all points greater than x_0 in the interval.

If these two signs are the same, there is a consistent concavity, and f can be said to have that concavity throughout the interval.

Most of the time, if $f''(x_0) = 0$, f'' will **change sign** at x_0. Such a point is called an **inflection point**.

Convexity. A figure S is said to be **convex** if, when-

ever two points p_0 and p_1 both belong to S, the entire line segment joining p_0 and p_1 is contained in S. This simple characterization has powerful consequences. One consequence is that **convex figures have tangents that stay on one side of the figure**.

Not surprisingly, that terminology of concavity and convexity are related. If a function f is concave upward on an interval I, the portion of the xy plane with $x \in I$ and $y > f(x)$ is a convex set.

This can be proved using the **Mean Value Theorem**. Suppose $x_0 < x_1$ in an interval where $f''(x) \geq 0$. The slope of the **chord** joining the points on the graph of $y = f(x)$ with these values of x is equal to the value of f' between x_0 and x_1. Thus, the slope of the tangent at $x = x_0$, the slope of the chord, and the slope of the tangent at $x = x_1$ are an **increasing** sequence. However, the height of the portion of a line to the right of a point increases with the slope. This shows that $f(x_1)$ is greater than the point with $x = x_1$ on the tangent at $x = x_0$. Since this is true for all x_1 in the interval, the curve is above the right side of any tangent line. A similar argument shows

the curve to be above the left side of every tangent line. Putting these parts together, the curve is above all tangent lines.

To show that chords lie above the curve, you show that the slope of the chord is an increasing function when the curve is concave upward. The **leading special case** has $x_0 = 0$ and $f(x_0) = 0$, so that the slope of the chord is $f(x)/x$. The derivative of this expression is

$$\frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - \frac{f(x)}{x}}{x}.$$

For $x > 0$ both numerator and denominator of the last expression are positive when f is concave upward.

The second derivative test. For some reason calculus textbooks love this topic. They convince students that second derivatives **must** be computed in every problem involving maxima or minima. This is fine when the second derivative is easy to find. However, if some effort will be required to find the derivative, one should ask whether anything useful will be learned before spending time on the calculation. In particular, the test is only meaningful if the **correct** second derivative is used. If you are going to make a mistake, you shouldn't do the work.

You probably won't believe me, but I will claim that you almost **never** need the second derivative test. The proof of this claim is contained in the proof of the second derivative test itself. When you see what the test really tells you, you will see that you often know that without doing the calculation.

Here is the test. Suppose the $x = c$ is a critical point of f, that is $f'(c) = 0$. Suppose also that $f''(c) > 0$. Then $f'(x)$ is an increasing function **at** $x = c$, and hence on some interval **near** $x = c$. Since f' is increasing and $f'(c) = 0$, it must be the case on this interval that $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$. This says that f is decreasing for smaller x and increasing for larger x, which says that $f(x) \geq f(c)$ for all x in the interval. Thus a positive second derivative at a critical point says the the critical point is a relative minimum. To remember this, recall the the positive second derivative says that the curve will be above the horizontal tangent at the critical point (just like $y = x^2$ at the origin). In **the same way**, one

shows that a negative second derivative at a critical point says that the critical point is a relative maximum. If the second derivative is zero at the critical point, the test is **inconclusive**.

At best, the test allows the **relative** or **local** properties of a critical point to be identified without consideration of any **global** properties.