Optimizing models. In Section 4.5, the techniques for finding maxima or minima of functions are combined with the construction of mathematical models (from Section 2.3). The details of constructing and presenting the model are often rushed in the eagerness to get to the Calculus, but this is poor practice. Although textbook problems often appear artificial, they can provide good practice in the the careful construction of models that allows mathematical methods to apply to practical problems.

Here is the process:

Step 1: Declare your variables. Everything mentioned in the problem should be identified and given a name. Algebra and Calculus have traditionally used single letters to name variables, but you can depart from this convention if you are clear about your intent. In most cases, the problem deals with **measurements**, but the mathematics requires **numbers** To connect the two, there should be **units of measurement** that are **always** used when when describing the problem and its answer in words, although the mathematics deals only with numbers. If appropriate, include a figure.

Step 2: Describe all relations between the variables. Also note any **feasibility conditions**. In many cases, some measurements need to positive in order to be feasible in the process being described. Such conditions must be identified early.

Step 3: Select an **independent variable** and express other variables in terms of it. If the problem can be solved by this method, you must be able to do this. (There is a corresponding theory of maxima and minima of functions of several variables, but it requires tools that are beyond the scope of this course.) For an optimization problem, there will be a single **objective** whose maximum or minimum value is sought in the problem. You should have expressed as a function of the independent variable.

Step 4: Find the extreme values of the function giving the objective in terms of the independent variable. Find the values of all variables when the objective takes an extreme value, and use these values to characterize the model when the objective takes an extreme value.

We illustrate this for some exercises from the textbook.

Exercise 4. Here is the statement of the problem: “By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, an open box may be made. If the cardboard is 15 in. long and 8 in. wide, find the dimensions of the box that will yield the maximum volume.

Some variables are the **length** l, **width** w and **depth** d of the box, measured in inches, which are needed to describe the box. The **volume** V of the box (in cubic inches) should appear, as well as the side of the square cut from each corner. Since all cuts are identical squares, only the **side** s of this square, in inches, is needed.

Here are the relations between these quantities:

$$\begin{aligned} l + 2s &= 15 \\ w + 2s &= 8 \\ d &= s \\ V &= lwd \end{aligned}$$

A box is formed if $l \geq 0$, $w \geq 0$ and $d \geq 0$.

The independent variable should be s. The first three equations give l, w and d directly. Then these can be combined to give V. Since we are trying to maximize V, this is the objective.

The expression for V in term of s is

$$V = (15 - 2s)(8 - 2s)s = 120s - 46s^2 + 4s^3$$

for $0 \leq s \leq 4$.

Differentiating,

$$\begin{aligned} \frac{dV}{ds} &= 120 - 92s + 12s^2 \\ &= 4(30 - 23s + 3s^2) \\ &= 4(6 - s)(5 - 3s) \end{aligned}$$

The critical values are $s = 0$, $s = 6$, and $s = 5/3$

Exercise 9. Postal regulations specify that a parcel sent by parcel post may have a combined length and girth of no more than 108 in. Find the dimensions of a rectangular package that has a square cross section and the largest volume that may be sent through the mail.

Exercise 11. Rework exercise 9 if the package is a cylinder with circular cross section.

Exercise 10. A production editor decided that the pages of a book should 1 in. margins at top and bottom and $\frac{1}{2}$ in. margins on the sides. She further stipulated that each page should have an area of 50 square inches. Find the page dimensions that will result in the largest printed area.

Exercise 16. An apple orchard has an average yield of 36 bushels of apples per tree if tree density is 22 trees per acre. For each unit increase in tree density, the yield decreases by 2 bushels. How many trees should be planted to maximize the yield?