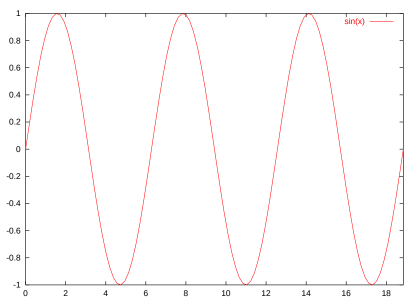
Old business. The general power rule can be derived from the calculus of the exponential and logarithm functions. This will explain why the exponent is decreased by 1 in the power rule with constant exponent while it remains unchanged for the exponential function with constant base.

If $f(x) = x^n$, we may write $f(x) = e^{n \ln x}$. Then

$$f'(x) = e^{n \ln x} \frac{d}{dx}(n \ln x) = x^n \frac{n}{x} = nx^{n-1}.$$

-X-

Consider this graph.



Trigonometry. The functions $\sin t$ and $\cos t$ describe the process of drawing a circle. Here $\cos t$ is the x-coordinate and $\sin t$ the y coordinate of a point on

the unit circle with equation $x^2 + y^2 = 1$, where t is supposed to be an intrinsic quantity known as **angle** that describes the location of the point on the circle.

One way to measure angles is to use the length of the arc on the circle. Another way is to use the area of the sector formed by the arc and the radii to the endpoints of the arc. Unfortunately, both of these approaches require calculus that has not yet been described in this course. Another approach is based on the idea of rotation as a **rigid motion**. Again, the details are missing in what has been done in this course. The results are **true** and **can be proved**. but our description of the functions will rely on an intuition about angles.

Periodicity. When you draw a circle, you eventually get back to your starting point. Since continued drawing gives no new points on the curve, one usually stops at this point. However, nothing forces you to stop, and you could continue indefinitely wrapping a number line around the circle. The point where you first meet your starting point defines a **period** of the functions describing the coordinates of the point on the circle. Although described initially only for positive value of the angle, it is easy to use periodicity to discover how negative angles should be interpreted. The first slide began with a graph showing two periods of $y = \sin(x)$.

Since many graphing programs give independent scaling of the axes, all graphs of $y = A \sin bx$ would look the same. Moreover, the graphs of $y = \sin x$ and $y = \cos x$ differ only by a horizontal translation.

A change of unit for angular measurement also has the effect of changing the scale on the x axis, or equivalently multiplying x by a constant before evaluating the function. As with exponentials and logarithms, several scales are in common use (most calculators allow **three** choices), but calculus selects a **natural** measure that leads to the best differentiation formulas.

The graph shows scales on the horizontal and vertical axes that play different roles: every horizontal distance can be measured with respect to the **period** of the function (the **smallest positive** number p such $f(x + p) = f(x)$); and vertical distances are measured with respect to the **amplitude** (the distance from the central value to the maxima or minima), which is

1 for $\sin x$.

Calculus. The graph of $y = \sin x$ **looks smooth**, suggesting that this function is differentiable. Investigating the possibility of computing the derivative of $\sin x$ at $x = 0$ already reveals the need for

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Since a change of angular measurement is equivalent to replacing $\sin x$ by $\sin bx$ for some b, and

$$\lim_{x \to 0} \frac{\sin bx}{x} = \lim_{x \to 0} b \frac{\sin bx}{bx} = b \lim_{x \to 0} \frac{\sin x}{x},$$

the value of this limit depends on the scale of measurement, though the **existence** of the limit does not.

Once this limit is shown to exist, enough other limits can be calculated to show that $\sin x$ is differentiable everywhere. First, consider

$$\begin{aligned}\frac{1-\cos x}{x} &= \frac{1-\cos^2 x}{x(1+\cos x)} \\ &= \frac{\sin^2 x}{x(1+\cos x)} \\ &= (\sin x)\frac{\sin x}{x}\frac{1}{1+\cos x}.\end{aligned}$$

Each factor in the last expression has a limit, and one of then has limit zero, so we have

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

To differentiate $\sin x$, start with the **addition formula** for this function

$$\sin(x+h) = \sin x\cos h + \cos x\sin h.$$

Subtracting $\sin x$ and grouping terms gives

$$\sin(x+h) - \sin x = \sin x(\cos h - 1) + \cos x\sin h.$$

Dividing by h and selecting the most useful of distributing this factor gives

$$\frac{\sin(x+h)-\sin x}{h} = \sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}.$$

Each piece has a limit as $h \to 0$, so

$$\frac{d}{dx}(\sin x) = \left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cos x.$$

The best unit of measurement is one for which

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

Such a unit is one that measures angles by the length of the arc on the unit circle, so the measure of **once around** the unit circle is 2π. This unit of measurement is called **radian measure**. When using your calculator for exercises in this course, you **must** select that mode. All trigonometric functions have period 2π.

Radian measure of important angles. Since the whole circle is 2π radians, a semicircle, which is the measure of a **straight angle**, is π radians. Half of a straight angle is a **right angle** whose measure is thus $\pi/2$. The sum of the interior angles of a triangle is a straight angle, so an **isosceles right triangle** has one right angle of measure $\pi/2$ and two acute angles, each of size $\pi/4$. The **equilateral triangle** has three equal angles, each of size $\pi/3$. If you bisect one of the angles to get an angle of $\pi/6$, that angle bisector must be perpendicular to the opposite side, and it divided the original triangle into congruent parts, so the angle bisector is also a median and a perpendicular bisector of the opposite side. These examples give some angles whose trigonometric functions can be found exactly.

The above description required only a little bit of Euclid to give a description that used only radian measure. Since the **degree** measure of these angles are also known, any of these values can be used to relate the two measures. In particular, the straight angle of π radians is 180°.

What does "co" mean? Trigonometric functions occur in pairs: sine and cosine; tangent and cotangent; secant and cosecant. For acute angles, these are ratios of sides of a right triangle, and "co" indicates the ratio that would be obtained by using a particular definition for the **other** acute angle of the triangle. Since the sum of the acute angles of a right triangle is $\pi/2$, this "other" angle is the **co**mplement, and the angle is $\pi/2 - x$. In particular,

$$\cos x = \sin\left(\frac{\pi}{2} - x\right),$$

and

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) \\ &= \cos\left(\frac{\pi}{2} - x\right) \cdot \frac{d}{dx}\left(\frac{\pi}{2} - x\right) \\ &= -\cos\left(\frac{\pi}{2} - x\right) \\ &= -\sin x. \end{aligned}$$

Two more functions. The complementary angle trick above means that the only functions that need to be differentiated directly are

$$\tan x = \frac{\sin x}{\cos x} \text{ and } \sec x = \frac{1}{\cos x}.$$

The quotient rule allows these to be differentiated now that we know the derivatives of $\sin x$ and $\cos x$. The results are:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

All that remains is to get some practice using these new formulas.