Old Business. Some problems from the exam should be reviewed before doing anything else. There were several versions of each question, and I will select one. Since the only difference between versions is in some of the numbers, the method used in one version will be valid for all.

Problem A1. To solve

$$(19)^{2x} - 20(19)^x + 19 = 0$$

introduce $u = 19^x$ so that the equation becomes

$$u^2 - 20u + 19 = 0.$$

The left side is $(u-1)(u-19)$, so the roots are $u = 1$ which corresponds to $x = 0$ and $u = 19$ which corresponds to $x = 1$.

Problem B2. To solve

$$\frac{10}{1+e^t} = 3$$

begin by introducing $v = e^t$ so that the equation becomes

$$\frac{10}{1+v} = 3.$$

Now, clear denominators to obtain

$$10 = 3 + 3v$$

which has the solution $v = 7/3$. This leads to $t = \ln(7/3) \approx 0.8473$.

Problem B6 This was taken from an assigned homework problem (Section 4.5, #5) using the same words (but a different side length) as the textbook. Although the language had some awkward quirks, the resulting difficulties should have been resolved in recitation. Here is the statement, with additional emphasis:

If an open box is made from a tin sheet 12 **in. square** by cutting out **identical squares** from each corner and bending up the resulting flaps, **determine the dimensions** of **the largest box** that can be made.

Some things said only in words are that the figure used to identify variables in the problem is a square 12 in. on each side. All distances should be measured in inches since that is the only unit of length mentioned in the problem. The dimensions of a box are **length**, **width**, and **depth**, but these must be determined from whatever distance is chosen as independent variable. The measure of the **size** of a box is **volume** since a box is something designed to hold things, and the volume of a box is the product of its dimensions. Since the box is determined by the squares cut from the corners, the independent variable should be the length of the side of that square. Call this s. When the flaps are folded up, s becomes the depth of the box. The length and width are the lengths of those flaps, which are the original 12 diminished by s at each end, so they are $12 - 2s$. The objective is therefore

$$V = s(12 - 2s)^2,$$

and

$$\begin{aligned} \frac{dV}{ds} &= (12 - 2s)^2 + (s)(2)(12 - 2s)(-2) \\ &= (12 - 2s)(12 - 6s) \end{aligned}$$

The feasible values of s, which have all sides of box nonnegative, are $0 \le s \le 6$, and $V = 0$ at both ends of this interval. The maximum will be taken on at

an interior critical point, and the only such point is $s = 2$. Substituting this into the expressions for the dimensions, we find that length and width are each 8 and depth is 2.

Problem B7 A paraphrase of part of the given information says that:

The domain of f consists of all real numbers except $x = 1$.

As x goes from $-\infty$ to 0, $f(x)$ decreases from 0 to -3.

As x goes from 0 to 1, $f(x)$ increases from -3 to $+\infty$.

As x goes from 1 to ∞, $f(x)$ increases from $-\infty$ to 0.

To guide the construction of the graph, a figure should be prepared as scratch work in which all values excluded by these conditions are shaded.

There is only one inflection point, concavity changes only at $x = -2$ and at the vertical asymptote. Since the slope of the curve is 0 at $x = -\infty$ and negative at $x = -2$, the slope is decreasing on this interval, so the second derivative is negative there. From $x = 0$ to $x = 1$, the slope increases from 0 to $+\infty$, and from $x = 1$ to $+\infty$, the slope decreases from $+\infty$ to 0. The graph is concave upward for $0 < x < 1$ and concave downward outside this interval. The significance of this is that the curve is **above its tangent lines** where it is concave upward, and below its tangents when concave downward. Drawing some tangent lines and shading the **outside** of those lines allows the graph to be found as the edge of the shaded region. This will be illustrated in hand-drawn slides. (Unfortunately, I don't have any way to transcribe this to a computer easily, so it won't be posted.)

More old business. The idea of the integral as antiderivative and the notation for it given in Section 6.1 should be reviewed.

Integration by substitution. Although the **chain rule** is one of the more troubling parts of the differential calculus, its counterpart in the integral calculus is helped by a notation that encourages its use.

Since

$$\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big) \cdot g'(x),$$

this may be written as an integral calculus statement:

$$\int f'\big(g(x)\big) \cdot g'(x)\, dx = f(u) = \int f'(u)\, du,$$

where $u = g(x)$.

If we were given the integral in terms of u and the definition $u = g(x)$, all that we would need to do to obtain the integral in terms of x is to make this substitution for **every** u in the expression — including replacing du by $g'(x)\, dx$.

Unfortunately, substitution is usually used the other way around. That is, the integral in terms of x is evaluated by **recognizing it** as the result of such a substitution. This if done by discovering an expression $g(x)$ so that, **except for a multiplicative factor** of $g'(x)$, the given integrand can be written as a function of $g(x)$.

Sometimes, it is not necessary to do an explicit substitution step to rewrite the integral in terms of a new variable. Just knowing that the method applies can help to discover the value of the integral up to a constant factor. The use of the chain rule in checking that result may allow for simplification that is not possible is factors arising in different parts of the computation are kept apart by an explicit substitution.