The theory of Area. Newton was born in 1642 and Calculus was formulated during his lifetime, but fragments that were eventually absorbed into Calculus and used later to provide its rigorous foundation. In particular, a method of computing area using what are now called Riemann sums (after Bernhard Riemann, 1826 – 1866) was known in some form to Archimedes (c. 287 – 212 B.C.) Without some of the history, the significance of the facts can get lost. The danger of this is greater when 2000 years of a subject is run together as background for introducing the contribution of Calculus to this subject.

If you have a rectangle with integer sides a and b, it can be cut into ab squares of side 1. Since area measures how big a plane figure is, with the unit square as the basic unit, this rectangle has area ab. This formula extends to the case in which a and b are arbitrary positive real numbers.

Areas of figures like parallelograms, triangles, or even general polygons can be found by cutting one figure into pieces that are reassembled to give the other, but this doesn't seem to apply to figures with curved sides.

The missing ingredient is a type of limit. Archimedes describe a method for finding the area of a circle that was good enough to allow him to show that $\pi < 22/7$, and can now be used with a modest amount of programming to allow you to compute twice as many digits of π as your calculator has memorized. The method finds the areas of **inscribed** and **circumscribed** polygons of 2^n sides using a formula that relates these values for one value of n to those for the next value of n. The details of the computation are specific to the circle, but it leads to the general idea of squeezing the figure between two families of objects whose size can be found while the difference between a larger and a smaller area can be made arbitrarily small.

Riemann sums. The contribution of Riemann was to describe such a construction that applies to the area under any graph of a function $f(x)$. (The word "under" suggests that the function should be positive, but this is only use to give a convenient **picture** of the process. Riemann sums can be defined for arbitrary functions and have an interpretation using **signed area**.)

The idea is to divide the domain $[a, b]$ into n equal pieces of length $\Delta x = (b - a)/n$. On the piece $I_i = [a + (i - 1)\Delta x, a + i\,\Delta x]$, for $1 \le i \le n$, select a number x_i in I_i and use $f(x_i)$ to estimate the height of the graph of $f(x)$ over I_i. The strip under the graph of $f(x)$ over the interval I_i is essentially a rectangle of width Δx and height $f(x_i)$. The are can then be **estimated** by

$$\left(\sum_{i=1}^{n} f(x_i)\right)\Delta x. \qquad (R)$$

Formula (R) is a Riemann sum associated with $f(x)$ on $[a, b]$. If we allow $n \to \infty$, we can investigate if these sums have a limit. If a limit (for all allowable choice of the x_i) exists, it is called the **definite (Riemann) integral** of $f(x)$ on $[a, b]$, and denoted

$$\int_a^b f(x)\,dx.$$

The relation of this to the indefinite integral described earlier in the chapter will be the subject of the next lecture.

The limit involved in the Riemann integral seems very awkward since for each n there are many choices of the x_i. However, for any reasonable function, one can imagine choosing the x_i to **minimize** (respectively **maximize**) $f(x)$ on I_i. This choice leads to a **lower sum** (respectively, an **upper sum**) with all other sums on n intervals having value between these extremes. Existence of the integral is guaranteed if the difference between upper and lower sums approaches zero as $n \to \infty$. The result that makes this useful is that this happens **for all continuous functions** $f(x)$.

Increasing functions If $f(x)$ is an increasing function, the smallest value of $f(x)$ on I_i is found by taking x_i to be the **left endpoint** $a + (i - 1)\Delta x$. Similarly, the largest value has $x_i = a(i\,\Delta x$ From this, it follows that the difference between upper and lower sums is

$$\big(f(b) - f(a)\big)\Delta x = \frac{\big(f(b) - f(a)\big)\big(b - a\big)}{n}$$

which is a **constant** multiple of $1/n$.

Not only do Riemann sums **converge** to the definite integral, but a precise estimate of the difference allows the integral to be **approximated** by the sum. The numerical integration routine on your calculator combines carefully selected Riemann sums to give an approximation that differs from the integral of a sufficiently smooth function by a constant multiple of n^{-4}, so that modest values of n can lead to useful approximations.