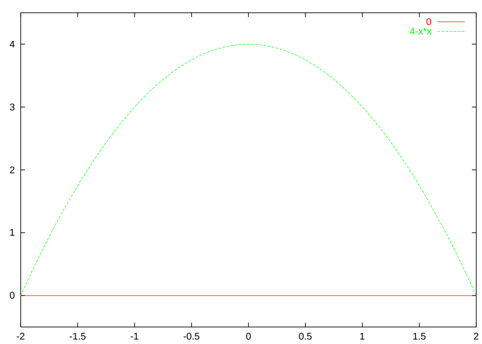
The Fundamental Theorem of Calculus. WOW! What a name for a theorem! What could be that big! In a way, the result has been anticipated: it says, "**integrals are integrals**". That is, the definite integral of $f(x)$ on an interval $[a, b]$, defined as a limit of a sum, can be computed from an indefinite integral $F(x)$, defined by $F'(x) = f(x)$ as $F(b) - F(a)$. Note that, although $G(x) = F(x) + C$ also satisfies $G'(x) = f(x)$, the computation of the definite integral isn't affected since $G(b) - G(a) = F(b) - F(a)$.

An example. Consider the portion of the parabola $y = 4 - x^2$ above the x axis.



It is easily seen that $0 \le 4 - x^2 = (2 - x)(2 + x)$ for $-2 \le x \le 2$, so the area under this curve is

$$A = \int_{-2}^{2} 4 - x^2\,dx$$

The fundamental theorem says that if we take **any** $\int 4 - x^2\,dx$, for example $F(x) = 4x - \tfrac{1}{3}x^3$, then

$$A = F(2) - F(-2) =$$
$$= \left(4(2) - \tfrac{1}{3}(2)^3\right) - \left(4(-2) - \tfrac{1}{3}(-2)^3\right) =$$
$$= \frac{16}{3} - \frac{-16}{3} = \frac{32}{3}.$$

A special notation. What do you write when you use the fundamental theorem in this way? It is awkward to try to find for what you are doing in the middle of a computation, so we need some way to remember the interval $[a, b]$ while writing $F(x)$, so that all the ingredients will be available when we assemble $F(b) - F(a)$. Here is the standard notation for this.

$$\text{If } F(x) = \int f(x)\,dx,$$

$$\text{then } \int_a^b f(x)\,dx = F(x)\Big|_a^b = \big(F(b)\big) - \big(F(a)\big).$$

When using this, you write the **actual expression** for $F(x)$ that you obtain as an indefinite integral wherever F appears in this description. I have also written parentheses around $F(b)$ and $F(a)$ to remind you that these quantities need to be found completely from the definition of F before beginning the **arithmetic** simplification. Most mistakes in this work involve losing track of the sign of a term. The extra care of a **structured** computation can improve the chance of getting a correct answer.

A paradox. On p. 481, you are invited to have a group discussion of the following facts:

(1) $F(x) = \int \frac{1}{x^2}\,dx = \frac{-1}{x}$ and $F(1) - F(-1) = -2$

(2) $\frac{1}{x^2} > 0$ where it is defined, so all Riemann sums of this function are positive.

This only appears to violate the Fundamental Theorem because we have been sloppy. Every theorem

has a hypothesis as well as a conclusion, and the existence of the definite integral is only guaranteed for **continuous** functions. There is a serious difficulty with $1/x^2$ near $x = 0$ and this value is **inside** the interval $[-1, 1]$. An integral of this function can only be expected to exist if both endpoints are on the same side of 0. The ability to evaluate F at the endpoints of the interval only correctly evaluates integrals that exist.

The idea of the proof of the fundamental theorem. We can use the notation of the fundamental theorem with a **variable upper endpoint**. That is

$$F(t) = \int_a^t f(x)\,dx$$

is a function of t such that: (1) $F(a) = 0$; (2) $F(t)$ gives the area under the graph of $f(x)$ between $x = a$ and $x = t$. If $f(x)$ is continuous on an interval I containing a, then $F(t)$ is defined on I **even if we don't have a formula for it**. Suppose we try to differentiate $F(t)$ (with respect to t). The definition of the derivative tells us to start with

$$F(t+h) - F(t).$$

This is a difference of areas where one area is contained in other. This should give the area **between** the outer and inner regions. (The difficulty here is that we used very special Riemann sums in or definition of the definite integral.) This difference is the area of the strip under the graph of $f(x)$ between $x = t$ and $x = t + h$. This can be written as h times an **average height** of the graph on this interval. If $f(x)$ is continuous, this must approach $f(t)$ as $h \to 0$.