Trigonometric Integrals. In lecture 18, we developed the differential calculus of trigonometric functions. The special limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

which requires that angles be measured in **radians**, was used to find that

$$\frac{d}{dx}\left(\sin x \right) = \cos x.$$

Then, the prefix "co" was found to mean "complement", so that

$$\cos x = \sin\left(\tfrac{\pi}{2} - x \right),$$

which led to

$$\frac{d}{dx}\left(\cos x \right) = -\sin x.$$

The quotient rule then gave

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

A similar analysis, or another use of the "co" trick gives the rarely used derivatives of $cot x$ and $\csc x$.

Armed with this information, the textbook pretends to give you some integration rules by inverting these differentiation formulas. This is only useful for dealing with integrals that are given to you in one of these unlikely forms.

Substitution. Integrals involving trigonometric functions evaluated at expressions other than the variable of integration can sometimes be found by a substitution that chooses a new variable equal to that expression. If the expression is **linear**, the substitution **will** work. For example,

$$\int \sin 2x \, dx = -\frac{1}{2} \cos 2x + C.$$

It is also possible to use a substitution in which the new variable contains a trigonometric function. Example 3 is

$$\int \frac{\sin x\, dx}{1+\cos x} = -\ln\left|1+\cos x\right| + C.$$

This was done by making the substitution $u = 1 + \cos x$.

Applying the same method to

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x}$$

leads to

$$\int \tan x\, dx = -\ln\left|\cos x\right| + C.$$

Something that arises often enough that it should get special mention is that the integral of any function of $\cos x$ timessin x cries out for the substitution $u = \cos x$. In particular,

$$\int \cos^n x \sin x\, dx = -\frac{1}{n+1}\cos^{n+1} x. \quad (n \neq -1)$$

Although some of these may be useful enough to be treated as integration formulas, the skill of recognizing them as **examples of substitution** is more useful.

A lucky discovery. It was possible to find $\int \tan x\, dx$ by a fairly natural substitution, but $\int \sec x\, dx$ does not lend itself to such a method. Other integrals of powers of $\sin x$ times powers of $\cos x$ can be found by a something resembling a **method**, but this one was originally found by a very indirect method until it was noticed that the result could be written as

$$\ln\left|\sec x + \tan x\right|.$$

Compound interest. We also fill in a topic skipped on our first treatment of the exponential function. The basic concept of **interest** is that if A allows B to use his money, B has gotten something of value even if he returns all the money that he borrowed. If we try to put a value on this **use** of money, it is reasonable that the value should be proportional to the amount of money involved (the **principal**), and at first, it **seems** reasonable that it should also be proportional to the time of the loan. This works well if the rules of the game demand that payment for the use of borrowed money (called **interest**) be paid on a fixed schedule. However, if a payment is missed, the borrower may agree to add the interest to the amount of the loan. Then, he may be led to the following analysis.

If I had declared the term of my loan to be half of the standard period, and charged half the rate for a half term, but failed to collect the payment when it was due, then the loan would have been increased by that amount for the second half of the term, so the total interest would be larger.

This is good! I could then cut that half-term in half and increase the payment even more; then cut that in half; and again . . .

This leads to a different way of looking at interest: instead of considering it to be a different kind of money than the principal of the loan, it should be the same kind of money, and it should be combined with the principal at the end of the term to get an **accumulated amount**. Then the formula $I = Pr$ for the interest at the end of one standard term changes to

$A = P(1+r)$ for the amount at the end of that term. Dividing the term in half as described above changes the amount at the end of one full term to

$$P\left(1+\frac{r}{2}\right)^2.$$

If the term were to be divided into n parts, with the rate divided by n on each part, then the amount at the end of a full term would be

$$P\left(1+\frac{r}{n}\right)^n.$$

As the number of parts n goes to infinity, the rate $x = r/n$ on each part is going to zero. Expressing the amount for a full term in terms of x gives

$$\left(1+\frac{r}{n}\right)^n = (1+x)^{r/x} = \left((1+x)^{1/x}\right)^r.$$

However,

$$\lim_{x\to 0}(1+x)^{1/x} = e,$$

so the effect of dividing the term into a very large number of intervals (called **continuous compounding**) is to cause the amount to be multiplied by e^r in one standard term.

Now, we have a formula that is easy enough to be used with arbitrary periods of time: with an **annual** rate of r and a time of t **years**, the amount is multiplied by e^{rt}.

This is a striking example of modeling. We set out only to interpret the effect of changing the term of loans at **simple** interest, but were led to **new** formulas that **forced** a particular type of formula to be used to describe all forms of interest. Note that the mathematical formula becomes simplest when expressed in terms of the **instantaneous** rate of interest, but the law (as described in the textbook) created an **effective annual rate** that is the annual rate of the simple interest model with a term of one year. Fortunately, as long as one accepts to exponential nature of compound interest, there are easy ways to convert between different ways of expressing the rate.