FALL 2008 FINAL EXAM

1. Find the following limits (5 points each), giving reasons for your answers. You may use any method from this course.

a. $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x} =$ $\underline{\quad -\frac{1}{8} \quad}$

$$\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)} \cdot \frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim_{x\to 2} \frac{(x+2)-2x}{x(x-2)(\sqrt{x+2}+\sqrt{2x})} = \cancel{\text{ }}$$

$$= \lim_{x\to 2} \frac{-\cancel{(x-2)}}{x\cancel{(x-2)}(\sqrt{x+2}+\sqrt{2x})} = \frac{-1}{2(2+2)} = -\frac{1}{8}$$

b. $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^{3x} =$ $\underline{\quad e^6 \quad}$

$$y = \left(1+\frac{2}{x}\right)^{3x} \qquad \ln y = 3x\ln\left(1+\frac{2}{x}\right)$$

$$\ln y = 3\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \qquad \lim_{x\to\infty} \ln y = 3\lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(\frac{-2}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= 3\lim_{x\to\infty} \frac{\frac{2}{1+\frac{2}{x}}}{1} = 3\cdot 2 = 6$$

So $\lim\limits_{x\to\infty} y = e^6$

c. $\lim\limits_{x\to 0} \dfrac{\sin(5x)-5x}{x^3} =$ $\underline{\quad -\frac{125}{6} \quad}$

$$\lim_{x\to 0} \frac{5\cos 5x - 5}{3x^2} = \lim_{x\to 0} \frac{-25\sin 5x}{6x} = \lim_{x\to 0} \frac{-125\cos 5x}{6}$$

$$= \frac{-125}{6}$$

2. Find the derivatives of the following functions (7 points each). You do not need to simplify your answers.

a. If $y = \tan(3x^2 + e)$ then $\frac{dy}{dx} =$ $6x \sec^2(3x^2+e)$

$$\frac{dy}{dx} = \sec^2(3x^2+e) \cdot (6x)$$

b. If $y = e^{\frac{x}{x+1}}$ then $\frac{dy}{dx} =$ $\dfrac{e^{\frac{x}{x+1}}}{(x+1)^2}$

$$\frac{dy}{dx} = e^{\frac{x}{x+1}} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right) = e^{\frac{x}{x+1}} \cdot \frac{(x+1)\cdot 1 - x}{(x+1)^2}$$

$$= e^{\frac{x}{x+1}} \left(\frac{1}{(x+1)^2}\right)$$

3. Find the following indefinite integrals (7 points each).

a. $\int t^2 \cos(1-t^3)\,dt =$ $-\frac{1}{3}\sin(1-t^3)+C$

$u = 1-t^3$

$du = -3t^2\,dt$

$t^2\,dt = -\frac{1}{3}\,du$

$\int \cos u \left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\sin u + C$

$= -\frac{1}{3}\sin(1-t^3)+C$

b. $\int \sqrt{x-1}\,dx =$ $\frac{2}{3}(x-1)^{3/2} + C$

$u = x-1$

$du = dx$

$\int \sqrt{u}\,du = \frac{u^{3/2}}{\frac{3}{2}} + C$

$= \frac{2}{3}(x-1)^{3/2} + C$

4. Calculate the following definite integrals (7 points each).

a. $\displaystyle\int_2^3 \frac{\ln(x)}{x}\,dx =$ $\frac{1}{2}\left((\ln 3)^2 - (\ln 2)^2\right)$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_{\ln 2}^{\ln 3} u\,du = \frac{u^2}{2}\Big|_{\ln 2}^{\ln 3} = \frac{1}{2}\left((\ln 3)^2 - (\ln 2)^2\right)$$

b. $\displaystyle\int_1^2 x\sqrt{x-1}\,dx =$ $\frac{16}{15}$

$u = x - 1$

$du = dx$

$x = u + 1$

$$\int_0^1 (u+1)u^{\frac{1}{2}}\,du = \int_0^1 \left(u^{3/2} + u^{\frac{1}{2}}\right) du$$

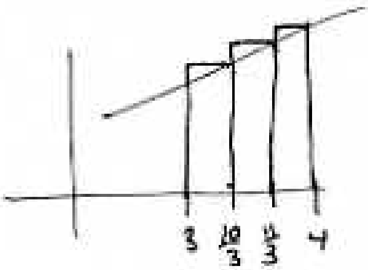
$$= \frac{u^{5/2}}{\left(\frac{5}{2}\right)} + \frac{u^{3/2}}{\left(\frac{3}{2}\right)} \Big|_0^1$$

$$= \left(\frac{2}{5} + \frac{2}{3}\right) = \left(\frac{6}{15} + \frac{10}{15}\right)$$

$$= \frac{16}{15}$$

5. (14 points) Estimate the area under the graph of $f(x) = x^2 + 5x$ from $x = 3$ to $x = 4$ using 3 equally spaced approximating rectangles and right endpoints. You may leave your answer as a sum. You will receive no credit for evaluating the integral exactly.

Approximate area:	$\frac{860}{27}$



$$\frac{1}{3}\left(f\left(\tfrac{10}{3}\right)+f\left(\tfrac{11}{3}\right)+f\left(\tfrac{12}{3}\right)\right) =$$

$$= \frac{1}{3}\left(\left(\tfrac{10}{3}\right)^2+5\left(\tfrac{10}{3}\right)+\left(\tfrac{11}{3}\right)^2+5\left(\tfrac{11}{3}\right)+\left(\tfrac{12}{3}\right)^2+5\cdot\tfrac{12}{3}\right)$$

$$= \frac{860}{27}$$ You don't have to multiply this out and add it up!

6. (15 points) Find the equation of the normal line to the curve described by

$$5x^2y + 2y^3 = 22$$

at the point $(2, 1)$. Any correct equation specifying this line is acceptable.[1]

Normal line:	$(y-1) = \frac{13}{10}(x-2)$

Implicit diff: $10xy + 5x^2y' + 6y^2y' = 0$

$$10\cdot 2\cdot 1 + 5\cdot 2^2\cdot y' + 6\cdot 1^2\cdot y' = 0$$

$$20 + 20y' + 6y' = 0$$

$$y' = -\frac{20}{26} = -\frac{10}{13}$$

Slope of normal line is $\frac{13}{10}$

[1]The normal line is perpendicular to the tangent line.

7. (14 points) A radioactive frog hops out of a pond full of nuclear waste in Oak Ridge, TN. If its level of radioactivity declines to 1/3 of the original value in 30 days, when will its level of radioactivity reach 1/100 of its original value? Note that this is an exponential decay problem.

# of days:	$30 \dfrac{\ln(100)}{\ln(3)}$

$$R(t) = R_0 e^{-kt}$$

$$\frac{1}{3} \cancel{R_0} = \cancel{R_0} e^{-k \cdot 30} \qquad \ln\left(\frac{1}{3}\right) = -k \cdot 30$$

$$-\ln(3) \qquad k = \frac{\ln(3)}{30}$$

$$R(t) = R_0 e^{-\frac{\ln(3)}{30} t}$$

$$\frac{1}{100} \cancel{R_0} = \cancel{R_0} e^{-\frac{\ln(3)}{30} t}$$

$$\ln\left(\frac{1}{100}\right) = -\ln(100) = -\frac{\ln(3)}{30} t$$

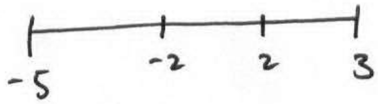
$$t = \frac{30 \ln(100)}{\ln(3)}$$

8. (14 pts) Let $f(x) = x^3 - 12x + 5$ on the interval $[-5, 3]$. Find the absolute maximum and minimum of $f(x)$ on this interval.

Absolute max:	21 at $x=-2$
Absolute min:	-60 at $x=-5$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

$$f'(x) = 0 \qquad x = \pm 2$$



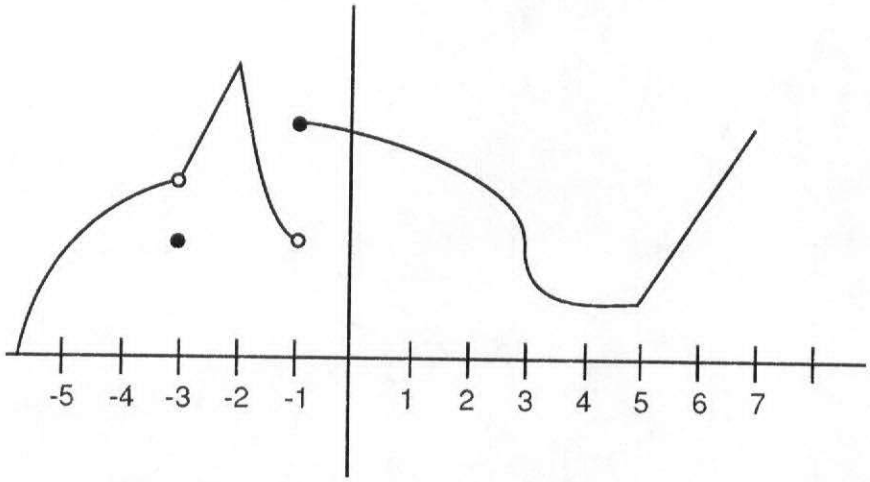
$f(-5) = -60$

$f(-2) = 21$

$f(2) = -11$

$f(3) = -4$

9. (14 points) The graph of $y = g(x)$ is given.
   a. For which values of x is $g(x)$ discontinuous? Don't worry about the endpoints at -6 and 7 in either part a or part b.
   b. For which values of x is $g(x)$ not differentiable?



Discontinuous	$x = -3$, $x = -1$
Not differentiable	$x = -3$, $x = -2$, $x = -1$, $x = 3$, $x = 5$

There is a vertical tangent at $x = 3$...

10. (14 pts) During the summer months, Terry makes and sells necklaces on the beach. Last summer, he sold the necklaces for \$10 each and his sales averaged 20 per day. He also found that for each \$1 increase in price sales drop by two per day. If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

Price:	\$13

This is from 4.7, so there won't be a problem like this on the spring 2009 exam.

$$P = \underbrace{(10+x)(20-2x)}_{\text{revenue}} - \underbrace{6\cdot(20-2x)}_{\text{cost}}$$

$$= (4+x)(20-2x)$$

$$P'(x) = (4+x)(-2) + (1)(20-2x) = -8-2x+20-2x$$

$$= 12-4x$$

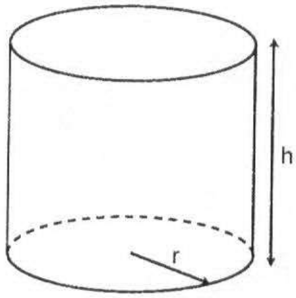
$P'(x) = 0$ when $x = 3$

For $x < 3$, $P'(x) > 0$ and for $x > 3$, $P'(x) < 0$ so

$P(x)$ has max at $x = 3$

Selling price should be \$13

11. (15 points) An open cylindrical can (without top) is to be constructed to hold 16π cubic cm of liquid. The cost of the material for the bottom is \$2 per cm^2, and the cost of the material for the curved surface is \$1 per cm^2. Find the radius and the height of the least expensive can. (The area of the curved surface is the circumference of the circle times the height.)



r=	2
h=	4

$C = 2\cdot\pi r^2 + 2\pi r h \qquad V = 16\pi = \pi r^2 h \Rightarrow h = \frac{16}{r^2}$

$C = 2\pi r^2 + 2\pi r\left(\frac{16}{r^2}\right) = 2\pi r^2 + 2\pi\left(\frac{16}{r}\right) = 2\pi\left[r^2 + \frac{16}{r}\right]$

$C' = 2\pi\left[2r - \frac{16}{r^2}\right]$

$C'(r) = 0$ when $2r = \frac{16}{r^2} \quad r^3 = 8 \quad r = 2$

$\boxed{\text{So } r=2,\ h=4}$

12. (15 points) Use linear approximation or differentials to find an approximate value for $\sqrt[3]{8.5}$.

Approx value:	$49/24$

$f(x) = x^{\frac{1}{3}} \quad f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

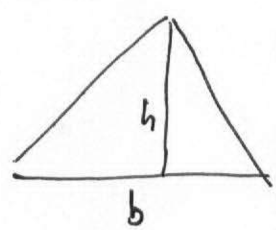
$f(x) \approx f(a) + f'(a)(x-a)$

$x = 8.5$, $a = 8$ $\Rightarrow \sqrt[3]{8.5} \approx \sqrt[3]{8} + \frac{1}{3}(8)^{-\frac{2}{3}}(8.5-8)$

$= 2 + \frac{1}{3}\left(\frac{1}{4}\right)(.5) = 2 + \frac{1}{24} = \frac{49}{24}$

13. (15 points) The altitude of a triangle is increasing at a rate of one ft/min while the area is increasing at a rate of 2 ft^2/min. At what rate is the base of the triangle changing when the altitude is 10 ft and the area is 100 ft^2?

Rate of change:	$-8/5$



$A = \frac{bh}{2}$

$\left.\begin{matrix} A = 100 \\ h = 10 \end{matrix}\right| \Rightarrow b = 20$

$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$

$2 = \frac{1}{2}\left(20 \cdot (1) + 10\frac{db}{dt}\right)$

$4 = 20 + 10\frac{db}{dt}$

$-16 = 10\frac{db}{dt}$

$\boxed{\frac{db}{dt} = -\frac{16}{10} = -\frac{8}{5}}$

or: $b = \frac{2A}{h}$

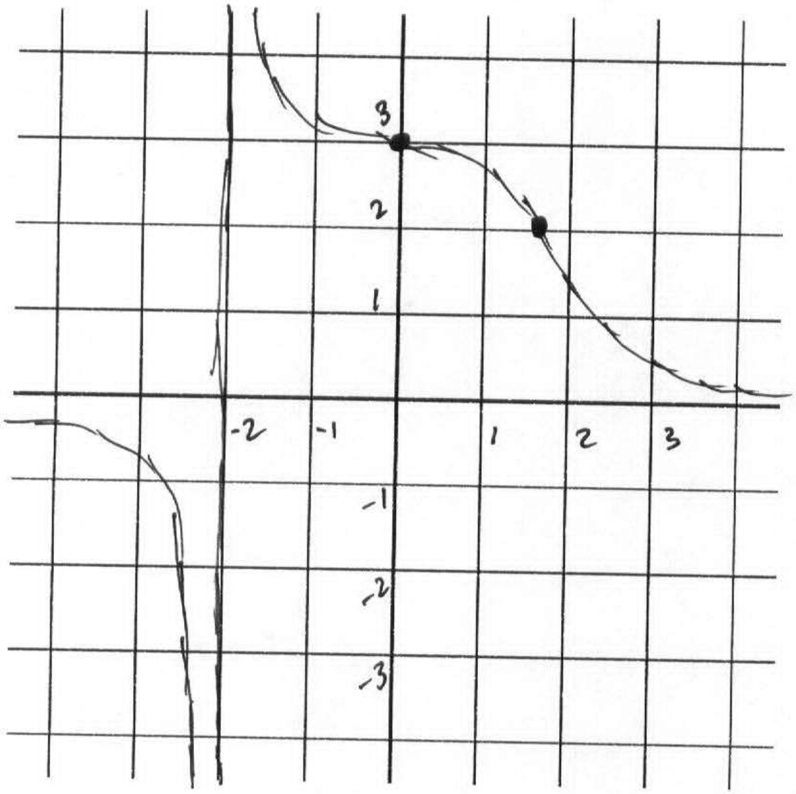
$\frac{db}{dt} = \frac{h \cdot 2\frac{dA}{dt} - (2A)\frac{dh}{dt}}{h^2}$

$= \frac{10 \cdot 2 \cdot 2 - 2 \cdot 100 \cdot 1}{100}$

$= \frac{40 - 200}{100} = -\frac{160}{100}$

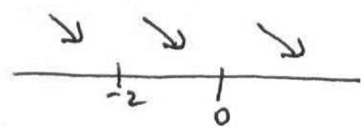
14. (15 points) Sketch the graph of the function $f(x) = 24/(x^3+8)$. For this function,

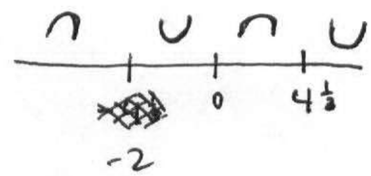
$$f'(x) = \frac{-72x^2}{(x^3+8)^2} \text{ and } f''(x) = \frac{288x(x^3-4)}{(x^3+8)^3}.$$



Horizontal asymptote(s):	$y = 0$
Vertical asymptote(s):	$x = -2$
Increasing:	NOWHERE
Decreasing:	$(-\infty, -2) \cup (-2, \infty)$
Concave up:	$(-2, 0) \cup (4^{\frac{1}{3}}, \infty)$
Concave down:	$(-\infty, -2) \cup (0, 4^{\frac{1}{3}})$
Relative max/min:	NONE
Inflections:	$x = 4^{\frac{1}{3}}$, $x = 0$

$\lim_{x \to \pm\infty} f(x) = 0$





$x = -2$ is a vertical asymptote, so it can't be an inflection.