FINAL F2009 ANSWERS

1. (6 points each) Find the following limits, giving reasons for your answers. You may use any method from this course.

a. $\lim_{x \to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} =$ $2\sqrt{3}$

$$\overset{L'HOP}{=} \lim_{x \to 3} \frac{1}{\frac{1}{2}x^{-\frac{1}{2}}} = \frac{2}{\frac{1}{\sqrt{3}}} = 2\sqrt{3}$$

OR MULTIPLY BY THE CONJUGATE, ETC

b. $\lim_{x \to 0} \frac{\cos 2x - 1}{x^2} =$ -2

$$\overset{L'HOP}{=} \lim_{x \to 0} \frac{-\cancel{2}\sin 2x}{\cancel{2}x} \overset{L'Hop}{=} \lim_{x \to 0} \frac{-2\cos 2x}{1} = -2$$

OR

$$\lim_{x \to 0} \frac{-2\sin 2x}{2x} = -2 \lim_{x \to 0} \frac{\sin 2x}{2x} = -2 \cdot 1 = -2$$

c. $\lim_{x \to \infty} x \sin\left(\frac{2}{x}\right) =$ $+2$

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} \overset{L'Hop}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{2}{x}\right) \cdot -\frac{2}{x^2}}{-\frac{1}{x^2}}$$

$$= \cos(0) \cdot (+2) = \underline{+2}$$

OR $z = \frac{2}{x}$

$\frac{1}{x} = \frac{z}{2}$

$$\lim_{z \to 0} \frac{\sin z}{\frac{z}{2}} = 2 \lim_{z \to 0} \frac{\sin z}{z} = 2$$

2. (9 points each) Find the derivatives of the following functions. You do not need to simplify your answers.

a. If $y = \cos^3 x \sin(x^5)$ then $\frac{dy}{dx} =$ ________________

$$y' = \frac{d}{dx}(\cos^3 x)\sin(x^5) + \cos^3 x \frac{d}{dx} \sin x^5$$

$$= 3\cos^2 x(-\sin x)(\sin x^5)$$

$$+ \cos^3 x \cdot \cos x^5 \cdot 5x^4$$

b. If $y = \frac{x \sin x}{1 + \ln x}$, then $\frac{dy}{dx} =$ ________________

$$\frac{(1+\ln x)(\sin x + x\cos x) - x\sin x\left(\frac{1}{x}\right)}{(1+\ln x)^2}$$

$$\parallel$$

$$\frac{x\cos x + x\cos x \ln x + \sin x \ln x}{(1+\ln x)^2}$$

3. (9 points each) Find the following indefinite integrals.

a. $\int \frac{t^2 + 5t + 1}{\sqrt{t}}\, dt =$ ______________

$= \int \left(\frac{t^2}{\sqrt{t}} + \frac{5t}{\sqrt{t}} + \frac{1}{\sqrt{t}} \right) dt$

$= \int \left(t^{3/2} + 5t^{1/2} + t^{-\frac{1}{2}} \right) dt$

$= \underline{\frac{t^{5/2}}{(5/2)} + 5\frac{t^{3/2}}{(3/2)} + \frac{t^{1/2}}{(1/2)} + C}$

b. $\int \sin(\cos x) \sin x\, dx =$ ______________

$u = \cos x$

$du = -\sin x\, dx$

$\int \sin u\, (-du) = \cos u + C$

$= \underline{\cos(\cos x) + C}$

4. (9 points each).

a. $\int_3^4 (1+e^x)^5 e^x \, dx=$ ________________________________ Do not try to simplify your answer!

$u = 1+e^x$

$du = e^x dx$

$$\int_{1+e^3}^{1+e^4} u^5 \, du = \frac{u^6}{6} \Big|_{1+e^3}^{1+e^4}$$

$$= \frac{1}{6}\left[(1+e^4)^6 - (1+e^3)^6\right]$$

b. $\int_2^3 x\sqrt{x-1}\, dx =$ ____________________

$u = x-1$

$du = dx$

$x = u+1$

$$\int_1^2 (u+1)u^{\frac{1}{2}} du = \int_1^2 (u^{3/2} + u^{1/2})\, du$$

$$= \frac{u^{5/2}}{(5/2)} + \frac{u^{3/2}}{(3/2)} \Big|_1^2$$

$$= \frac{2^{5/2}}{(5/2)} - \frac{1}{(5/2)} + \frac{2^{3/2}}{(3/2)} - \frac{1}{(3/2)}$$

5. (17 points) Let $f(x) = g(x^3 - 5)$. It is impossible to find $g(x)$, but a few values of $g(x)$ and $g'(x)$ are known: $g(1) = 2$, $g(2) = 5$, $g(3) = 7, g(4) = 2$, $g(5) = 11$, $g(6) = 13$, $g(7) = 21$, $g'(1) = 3$, $g'(2) = 2$, $g'(3) = 8$, $g'(4) = 10$, $g'(5) = 12$, $g'(6) = 21$ and $g'(7) = 23$.

a. Find $f(2)$. 7

b. find $f'(2)$. 24

$$f(2) = g(2^3 - 5) = g(3) = 7$$

$$f'(x) = g'(x^3-5) \cdot 3x^2$$

$$f'(2) = g'(\cancel{1})\cdot 12 = \cancel{3} \cdot 12 = \cancel{24}$$

3 8 96

6. (18 points) Find the equation of the tangent line to the curve described by

$$x^3y + xy^3 + x^2y - 2x^2 = -2$$

at the point (1, 0). Any correct equation specifying this line is acceptable.

Tangent line:	$(y - 0) = 2(x - 1)$

$$3x^2y + x^3y' + y^3 + 3xy^2y' + 2xy + x^2y' - 4x = 0$$

PLUG IN $x = 1, y = 0 \Rightarrow y' + y' - 4 = 0$

$y' = 2$

$$\frac{dy}{dx} = -\frac{3x^2y + y^3 + 2xy - 4x}{x^3 + 3xy^2 + x^2}$$

If you want to work out $\frac{dy}{dx}$ for all x + y

7. (18 points) The tangent line to $y = f(x)$ at $x = 2$ is given by $y = 7x + 3$.

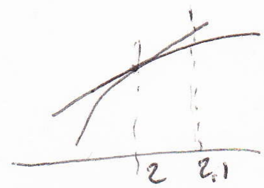
   a. (5 points) What is $f'(2)$? 7

   b. (5 points) What is $f(2)$? 17

   c. (6 points) Use linear approximation to approximate $f(2.1)$.

   $17 + 7 \cdot (.1) = 17.7$

   d. (2 points) If $f''(x) < 0$ for all x, is this approximation too large or too small? TOO LARGE



8. (17 pts) Find the absolute maximum and minimum of the function $f(x) = x^3 + 3x^2 - 9x$ on the interval $[-2, 2]$.

Absolute max:	(2,22)
Absolute min:	(1,-5)

$$f'(x) = 3x^2 + 6x - 9$$
$$= 3(x^2 + 2x - 3)$$
$$= 3(x+3)(x-1)$$

CRIT #S: $x = \cancel{-3}, 1$

$f(-2) = 22$

$f(1) = -5$

$f(2) = 2$

9. (6 points each)

a. Find $\frac{dy}{dx}$ if $y = x^{8x}$.

$\ln y = 8x \ln x$

$\frac{1}{y} y' = 8 \ln x + 8x \cdot \frac{1}{x}$

$y' = x^{8x}[8 \ln x + 8]$

b. Find $\frac{dy}{dx}$ if $y = \int_0^x \sin t^2 \, dt$.

$\frac{dy}{dx} = \sin x^2$

c. Find $\frac{dy}{dx}$ if $y = \int_0^{x^2} \sin t^2 \, dt$.

$\frac{dy}{dx} = \sin (x^2)^2 \cdot 2x$

$= (\sin x^4) \cdot 2x$

10. (18 points) A mad scientist sells radioactive bats to her friends. Experience tells her that she will sell 20 bats per month if she charges 30 dollars per bat and that each $2 decrease in price will result in four more sales per month. How much should she charge per bat to maximize her revenue?

Price per bat:	\$20

$R(x) = (30-2x)(20+4x)$

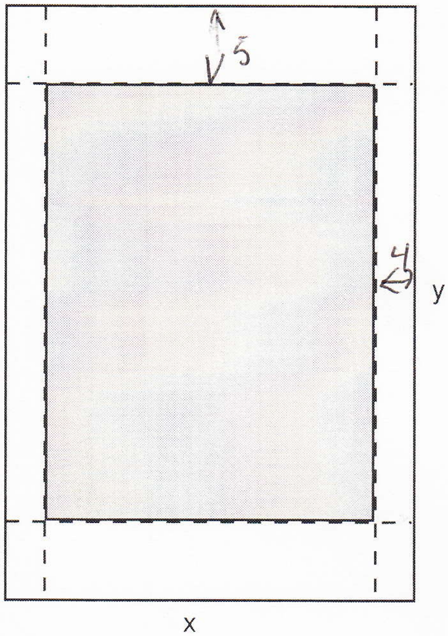
$R'(x) = -2(20+4x) + (30-2x)\cdot 4$

$= -40 - 8x + 120 - 8x$

$= 80 - 16x = 0 \Rightarrow x = 5$

She sells 40 bats at \$20 each.

11. (18 points) A rectangular poster is to contain 80 square inches of printed matter with 5 inch margins at the top and bottom and 4 inch margins at the sides. If posterboard costs 10 cents per square inch, what are the dimensions of the least expensive poster satisfying the requirements?



$$(x-8)(y-10)=80$$

$$y=\frac{80}{x-8}+10=\frac{10x}{x-8}$$

$$A=xy=\frac{10x^2}{x-8}$$

$$A'=\frac{(x-8)(20x)-10x^2\cdot 1}{(x-8)^2}$$

$$=\frac{20x^2-160x-10x^2}{(x-8)^2}=0$$

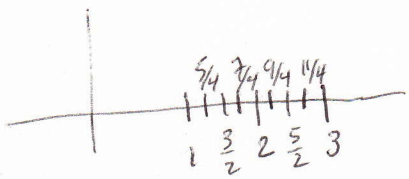
$$=\frac{10x^2-160x}{(x-8)^2}$$

$$\underline{\underline{x=16}}$$

$$y=\frac{10\cdot 16}{8}=20$$

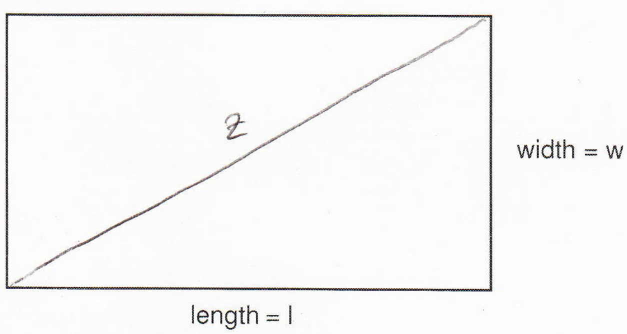
12. (18 points) 5. (14 points) Compute the value of the Riemann sum for the function $f(x) = x^2$ on the interval $[1, 3]$ using $n = 4$ and taking x_k^* to be the midpoint of the k^{th} interval in the partition. You can leave your answer as a sum of fractions.

Value:	



$$\frac{1}{2}\left[\left(\frac{5}{4}\right)^2+\left(\frac{7}{4}\right)^2+\left(\frac{9}{4}\right)^2+\left(\frac{11}{4}\right)^2\right]$$

13. (18 points) The length of a rectangle is decreasing at 4 in/min and its width is increasing at 5 in/min. How fast is the length of the diagonal changing when the length is 8 in and the width is 6 in?



$$z^2 = l^2 + w^2$$

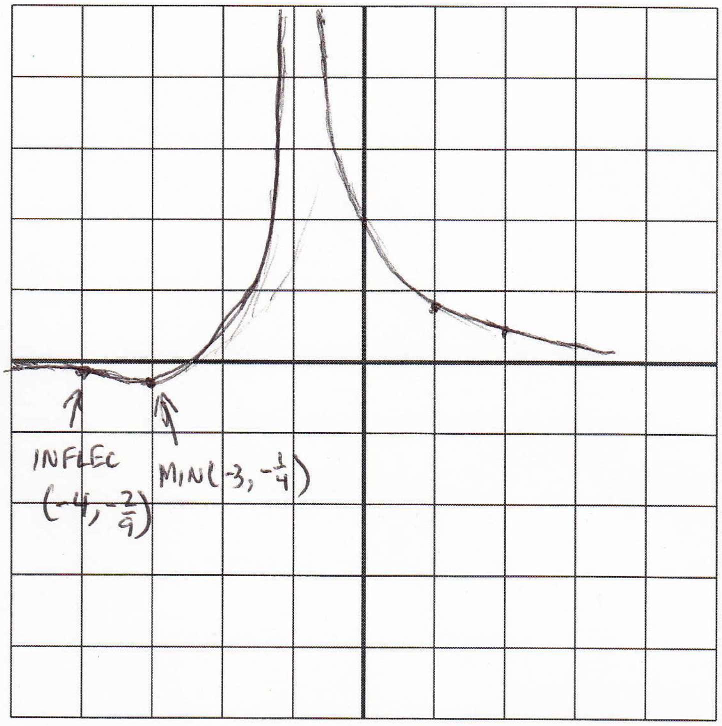
$$\cancel{2}z\frac{dz}{dt} = \cancel{2}l\frac{dl}{dt} + \cancel{2}w\frac{dw}{dt}$$

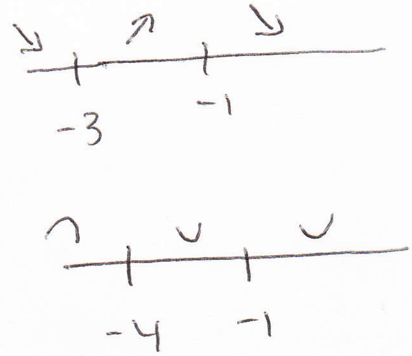
$$10\frac{dz}{dt} = 8\cdot(-4) + 6\cdot(5)$$

$$\frac{dz}{dt} = \frac{-2}{10} = -\frac{1}{5}$$

14. (18 points) Sketch the graph of the function $f(x) = \dfrac{x+2}{(x+1)^2}$. For this function,

$$f'(x) = -\frac{x+3}{(x+1)^3} \text{ and } f''(x) = \frac{2(x+4)}{(x+1)^4}.$$





Horizontal asymptote(s):	$y=0$
Vertical asymptote(s):	$x=-1$
Increasing:	$(-3,-1)$
Decreasing:	$(-\infty,-3)\cup(-1,\infty)$
Concave up:	$(-4,-1)\cup(-1,\infty)$
Concave down:	$(-\infty,-4)$
Relative max/min:	rel min $x=-3$ no rel max
Inflections:	$x=-4$