

- (13) 1. Find an equation for the tangent line to the graph of $y^2 = x^3 - 3xy + 3$ at the point $(-2, 1)$.

The first thing to do is to check that the values $x = -2$, $y = 1$ satisfy the given equation. They do.

Differentiating both sides of the equation with respect to x and remembering that y is a function of x , we get

$$2yy' = 3x^2 - 3xy' - 3y.$$

Solving for y' , we obtain

$$y' = \frac{3x^2 - 3y}{2y + 3x}.$$

At the point $(-2, 1)$, we have

$$y' = \frac{3 \cdot 4 - 3}{2 - 6} = -\frac{9}{4}.$$

An equation for the tangent is

$$y - 1 = -\frac{9}{4}(x + 2).$$

- (10) 2. Find equations for all vertical and horizontal asymptotes of the function

$$f(x) = \frac{3e^x + 5}{7e^x - 2}.$$

(All numbers used should be described by exact expressions, not decimal approximations. Thus you should write $\sqrt{2}$, not 1.414.)

Since e^x goes to 0 as x goes to $-\infty$, we have

$$\lim_{x \rightarrow -\infty} f(x) = \frac{3 \cdot 0 + 5}{7 \cdot 0 - 2} = -\frac{5}{2}.$$

Since e^x goes to ∞ as x goes to ∞ , we have

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{3 + \frac{5}{e^x}}{7 - \frac{2}{e^x}} = \frac{3 + 0}{7 + 0} = \frac{3}{7}.$$

Thus there are two horizontal asymptotes with equations $y = -5/2$ and $y = 3/7$.

The only way $f(x)$ can have a vertical asymptote is for the denominator $7e^x - 2$ to be 0. If

$$7e^x - 2 = 0,$$

then

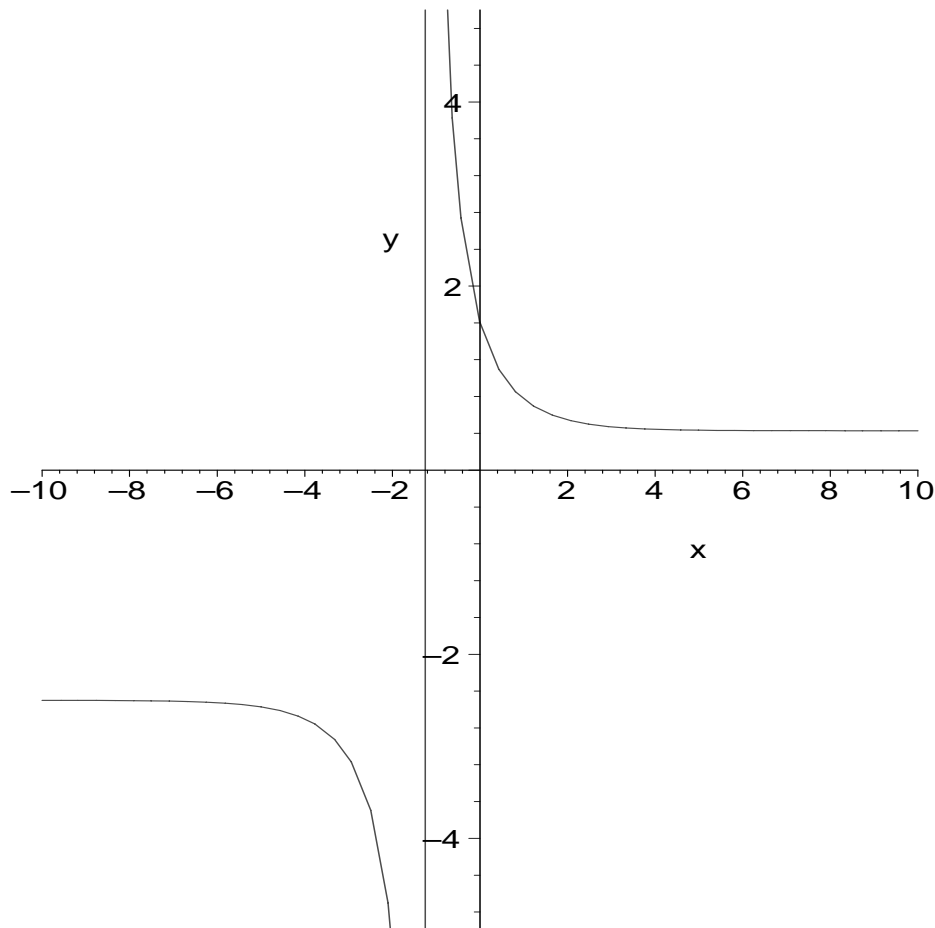
$$e^x = \frac{2}{7},$$

or

$$x = \ln(2/7).$$

There is one vertical asymptote with equation $x = \ln(2/7)$.

In this problem, the use of a graphing calculator can give good insight into the solution. Graphing the function on $[-10, 10]$ gives



from which you can see what you have to do. (Note: This graph was drawn using Maple, which automatically puts in the vertical asymptote!)

- (15) 3. At a certain time, the length of a rectangle is 5 feet and its width is 3 feet. At that same moment, the length is *decreasing* at 0.5 feet per second and the width is *increasing* at 0.4 feet per second.

What is the length of the diagonal at that time?

How fast is the length of the diagonal changing? Is this length increasing or decreasing?

Let x denote the length of the rectangle, y the width of the rectangle, and z the diagonal of the rectangle. The variables x , y , and z are all functions of time. In the following discussion, all derivatives are with respect to time and all distances are measured in feet.

At the moment in question we are given that $x = 5$, $y = 3$, $x' = -0.5$, and $y' = 0.4$. By the Pythagorean Theorem, $z^2 = x^2 + y^2$. Thus at the moment, $z^2 = 25 + 9 = 34$, so $z = \sqrt{34}$.

Differentiating the above equation, we have

$$2zz' = 2xx' + 2yy',$$

so

$$z' = \frac{2xx' + 2yy'}{2z} = \frac{xx' + yy'}{z}.$$

Thus at this moment

$$z' = \frac{5(-0.5) + 3(0.4)}{\sqrt{34}} = \frac{-1.3}{\sqrt{34}}.$$

Since z' is negative, the length of the diagonal is decreasing.

(10) 4. Suppose that $f(x) = \sqrt{2 + 7x^3}$.

Compute $f(1)$.

Compute $f'(1)$.

Use the linearization (differential, tangent line approximation) of f at $x = 1$ to estimate $f(1.08)$.

$$f(1) = \sqrt{2 + 7} = 3.$$

$f(x) = (2 + 7x^3)^{1/2}$. Thus

$$f'(x) = (1/2)(2 + 7x^3)^{-1/2}(21x^2) = \frac{21x^2}{2\sqrt{2 + 7x^3}}.$$

Hence

$$f'(1) = \frac{21}{6} = \frac{7}{2}.$$

The linearization of f at $x = 1$ is

$$L(x) = f(1) + f'(1)(x - 1) = 3 + \frac{7}{2}(x - 1).$$

The value of $L(1.08)$ is 3.28.

(5) 5. A friend runs up to you and excitedly explains that she has found a function g with the following properties:

g is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$g(0) = 1$ and $g(1) = 5$.

$$g'(x) \leq 3 \text{ for all } x \text{ in } (0, 1).$$

Explain why you doubt your friend's claim.

By the Mean Value Theorem, there must be a number c in $(0, 1)$ such that

$$g(c) = \frac{g(1) - g(0)}{1 - 0} = \frac{5 - 1}{1} = 4.$$

However, your friend asserts that $g'(x) \leq 3$ for all x in $(0, 1)$ and 4 is bigger than 3. Thus she must have made a mistake.

(24) 6. Suppose that $f(x) = \frac{x^2 + 3}{x^2 + x + 4}$.

(a) What is the domain of $f(x)$? Why?

Both the numerator and denominator of f are defined for all x . Thus the only way $f(x)$ could not be defined is if $x^2 + x + 4 = 0$. If this is so, then the quadratic formula says that

$$x = \frac{-1 \pm \sqrt{1 - 16}}{2} = \frac{-1 \pm \sqrt{-15}}{2}.$$

These values of x are not real numbers, so the domain of f is $(-\infty, \infty)$.

(b) What are $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$? Why?

If $x \neq 0$, then

$$f(x) = \frac{1 + \frac{3}{x^2}}{1 + \frac{1}{x} + \frac{4}{x^2}}.$$

Therefore, as $|x|$ goes to ∞ , $f(x)$ goes to 1. Thus both limits are equal to 1.

(c) Use calculus to find all relative extreme values of $f(x)$.

We must first find the critical numbers of f .

$$f'(x) = \frac{(x^2 + x + 4)(2x) - (x^2 + 3)(2x + 1)}{(x^2 + x + 4)^2} = \frac{x^2 + 2x - 3}{(x^2 + x + 4)^2}.$$

Thus $f'(x)$ is defined for all x and if $f'(x) = 0$, then

$$0 = x^2 + 2x - 3 = (x + 3)(x - 1),$$

which occurs when $x = -3$ or $x = 1$.

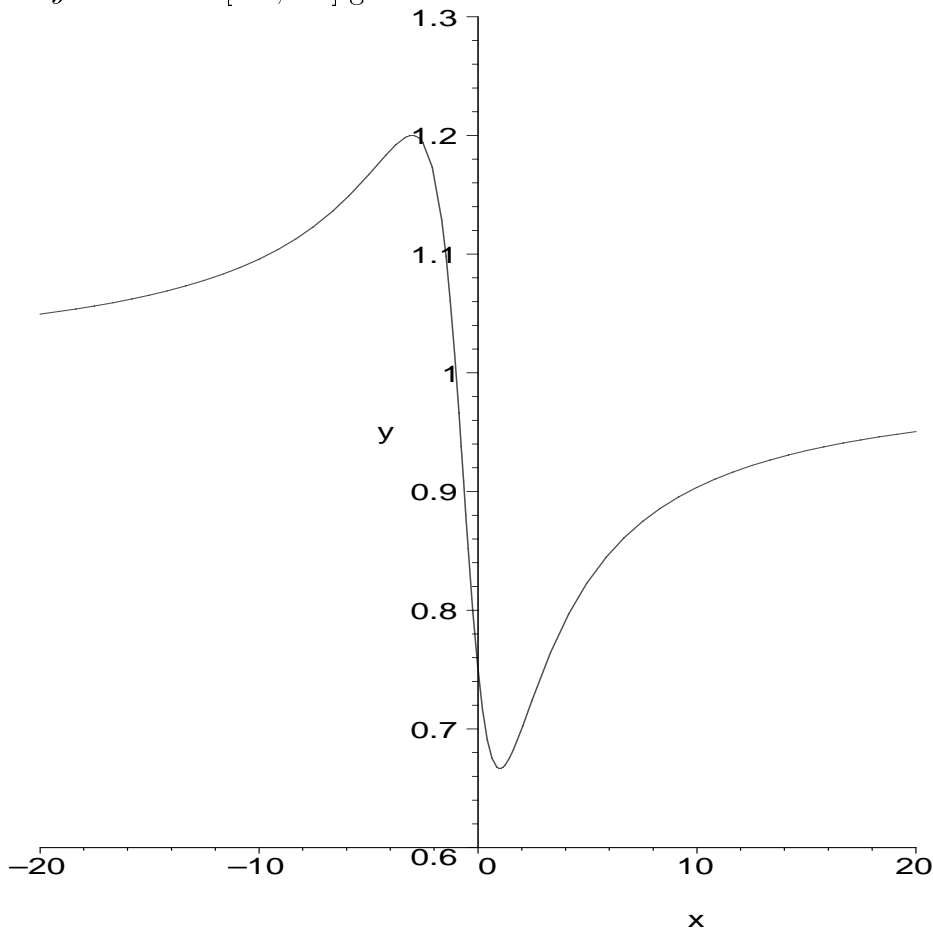
The value of $f(-3)$ is $6/5$ and $f'(x)$ changes sign from positive to negative at $x = -3$. Thus there is a relative maximum at $x = -3$.

The value of $f(1)$ is $2/3$ and $f'(x)$ changes sign from negative to positive at $x = 1$. Thus there is a relative minimum at $x = 1$.

(d) The *range* of a function is the collection of all possible values of that function. What is the range of f ? Explain your answer carefully.

By part (c), both $2/3$ and $6/5$ are values of f . By the Intermediate Value Theorem, any number between $2/3$ and $6/5$ is also a value of f . Thus the range contains $[2/3, 6/5]$. The value of $f(x)$ is close to 1 if $|x|$ is big and 1 is between $2/3$ and $6/5$. If there were any values of f outside $[2/3, 6/5]$, there would have to be other critical points. Therefore the range is $[2/3, 6/5]$.

This is another problem where a graphing calculator can give a good idea as to whether you have correctly analyzed the shape of the graph of this function. Graphing f on $[-20, 20]$ with values of y limited to $[0.6, 1.3]$ gives



You would still have to do the analysis above, but the picture would help you see whether you were on the right track.

- (15) 7. You wish to build a shed in the shape of a rectangular box with a square floor. The materials for the walls cost \$1 per square foot and the materials for the floor and roof cost \$2 per square foot. You want the shed to have a volume of 250 cubic feet. What should the dimensions of the shed be in order to minimize the cost of materials?

Let x denote the length of one side of the square base and let y denote the height of the shed. Since the volume of the shed must be 250 cubic feet, we have $x^2y = 250$ or

$$y = \frac{250}{x^2}.$$

The cost of materials for the floor and roof is $2 \cdot 2x^2$. The cost of materials for the four sides is $4xy$. Thus the total cost is

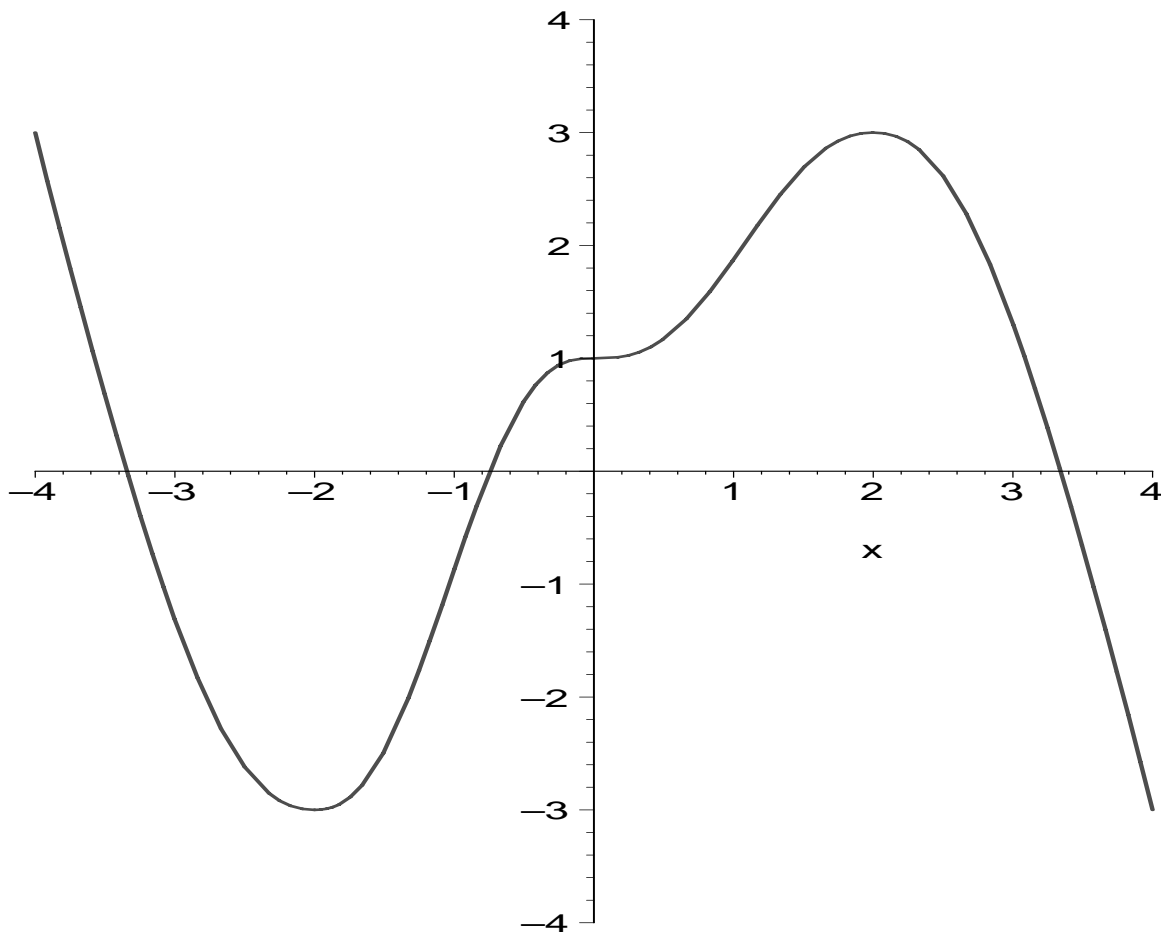
$$C = 4x^2 + 4xy = 4x^2 + 4x \left(\frac{250}{x^2} \right) = 4x^2 + \frac{1000}{x}.$$

Differentiating, we find

$$C' = 8x - \frac{1000}{x^2}.$$

Setting C' equal to 0, we get $8x^3 = 1000$ or $x^3 = 125$. From this we find that x is 5 feet and y is 10 feet. Physical considerations make it clear that this is a minimum.

- (8) 8. On the axes below sketch the graph of a function f with the following properties: The domain of f is $(-4, 4)$ and f is differentiable at all points in its domain. f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$. At $x = 0$ there is a horizontal tangent line and a point of inflection.



What is the total number of points of inflection of the function whose graph you have sketched?

There are three points of inflection in this graph corresponding approximately to $x = -1$, $x = 0$, and $x = 1$.