Final Exam 12/6/2005

Some Math 135 review problems

With answers

The following problems are based heavily on problems written by Professor Stephen Greenfield for his Math 135 class in spring 2005. His willingness to let these problems be used this semester is gratefully acknowledged. The problems are grouped by topic.
Answers, but not complete solutions, are given for most problems.

Definition of derivative

1. Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = x^2 - x$.

2. Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = \frac{1}{3x+4}$.

3. Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = \sqrt{1-2x}$.

Computations of limits

1. Evaluate the limits exactly. Give brief evidence supporting your answers which is not based on a calculator graph or on calculator computations.

a) $\displaystyle\lim_{x\to\infty} \frac{2x^2-5}{3x^2+1} = \frac{2}{3}$ b) $\displaystyle\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = 4$ c) $\displaystyle\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = 7$

d) $\displaystyle\lim_{x\to +\infty} \frac{2x^3-x+3}{x^3+2} = 2$ e) $\displaystyle\lim_{x\to 2^-} \frac{x^2-4}{|2-x|} = -4$ f) $\displaystyle\lim_{x\to 2^+} \frac{x^2-4}{|2-x|} = 4$

2. Find the equations of all vertical and horizontal asymptotes of $f(x) = \dfrac{\sqrt{x^2+4}}{x-3}$.

Ans.: The lines $y = 1$ and $y = -1$ are horizontal asymptotes and the line $x = 3$ is a vertical asymptote.

3. Find the equations of all vertical and horizontal asymptotes of $f(x) = \dfrac{3+5e^{-2x}}{7-e^{-2x}}$. Computations with exp and log should be simplified as much as possible; approximations are not acceptable.

Ans.: The lines $y = 3/7$ and $y = -5$ are horizontal asymptotes and the line $x = -\frac{1}{2}\ln(7)$ is a vertical asymptote.

Computations of derivatives

1. Find $\dfrac{dy}{dx}$.

a) $y = \dfrac{2x^2+5}{5x^3+1}$, $y' = \dfrac{(5x^3+1)(4x) - (2x^2+5)(15x^2)}{(5x^3+1)^2}$

b) $y = e^{-\sqrt{x^2-1}}$, $y' = -\dfrac{xe^{-\sqrt{x^2-1}}}{\sqrt{x^2-1}}$

c) $xy^3 = \cos(7x+5y)$, $\quad y' = -\dfrac{y^3 + 7\sin(7x+5y)}{3xy^2 + 5\sin(7x+5y)}$

d) $xe^y = \sin(xy)$, $\quad y' = \dfrac{-e^y + y\cos(xy)}{x(e^y - \cos(xy))}$

e) $y = x\ln(3x+5)$, $\quad y' = \ln(3x+5) + \dfrac{3x}{3x+5}$

f) $2x^3 + 5x^2y + y^3 = 2$, $\quad y' = -\dfrac{2x(3x+5y)}{5x^2+3y^2}$

g) $y = \ln(x^4 + 3x + 1)$, $\quad y' = \dfrac{4x^3+3}{x^4+3x+1}$

h) $y = \dfrac{3x^3-1}{3x^4+1}$, $\quad y' = \dfrac{(3x^4+1)(9x^2) - (3x^3-1)(12x^3)}{(3x^4+1)^2}$

i) $y = x^5\tan(3x)$, $\quad y' = 5x^4\tan(3x) + 3x^5\sec^2(3x)$

j) $y = (4x+3)\sqrt{x^3+7}$, $\quad y' = 4\sqrt{x^3+7} + \dfrac{3(4x+3)x^2}{2\sqrt{x^3+7}}$

2. Suppose $f(x)$ is a differentiable function satisfying $f'(1) = 3$ and $f'\left(\frac{\pi}{4}\right) = -2$, If $g(x) = f(\tan x)$, find $g'\left(\frac{\pi}{4}\right)$.

Ans.: 6

3. An equation of the tangent line to $y = f(x)$ when $x = 1$ is known to be $2x + y + 3 = 0$. Find $f(1)$ and $f'(1)$.

Ans.: $f(1) = -5$, $f'(1) = -2$

4. Suppose U is a differentiable function with $U(8) = 5$ and $U'(8) = 3$, and that $V(x) = U(x^3)$. What are $V(2)$ and $V'(2)$? Use this information to write an equation of the tangent line to $y = V(x)$ when $x = 2$.

Ans.: $V(2) = 5$, $V'(2) = 36$, the tangent is $y - 5 = 36(x-2)$.

5. A function is defined implicitly by the equation $x^2 - y^2 - 5xy + x + y = 12$. Find y' in terms of x and y. Find an equation for the line tangent to the graph of this function at the point $(-2, 1)$.

Ans.: $y' = \dfrac{2x - 5y + 1}{-1 + 2y + 5x}$, the tangent is $y - 1 = \frac{8}{9}(x+2)$.

Log/exp etc.

1. Find the range of $f(x) = e^{-2x} + e^{3x}$.

Ans.: The range is $[u, \infty)$, where $u = \dfrac{5 \cdot 2^{3/5} \cdot 3^{2/5}}{6} = 1.9601317\ldots$.

(There is a local minimum at $x = \ln(2/3)/5$ and the value of f there is u.)

Continuity & differentiability

1. Here $f(x) = \begin{cases} x+3 & \text{if } x \le -2 \\ \frac{1}{2}x^2 + A & \text{if } -2 < x \end{cases}$ where A is a constant to be determined. Find A so that $f(x)$ is continuous for all values of x. Sketch a graph of $y = f(x)$ using that value of A for $-4 \le x \le 2$. Is $f(x)$ differentiable at $x = -2$ using that value of A?

Ans.: $A = -1$. With this A, f is not differentiable at -2.

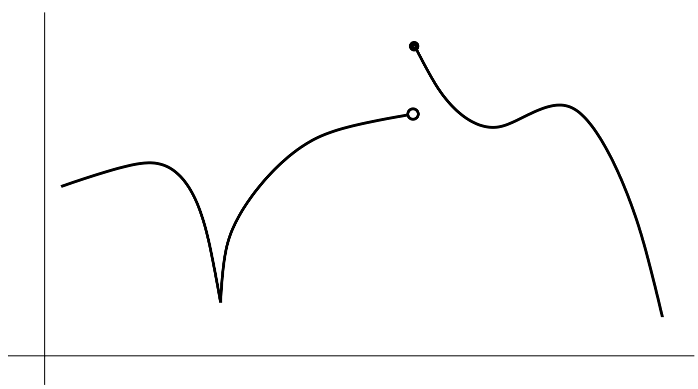
2. Here $f(x) = \begin{cases} Ax^2 - 1 & \text{if } x < -1 \\ x + B & \text{if } -1 \le x \le 1 \\ 2 & \text{if } 1 < x \end{cases}$ where A and B are constants to be determined. Find numbers A and B so that $f(x)$ is continuous for all values of x. Sketch a graph of $y = f(x)$ for $-3 \le x \le 3$.

Ans.: $A = B = 1$.

3. In this problem $f(x) = \begin{cases} 1 + x^2 & \text{if } x < -2 \\ A + Bx & \text{if } -2 \le x < 1 \\ x^2 & \text{if } x \ge 1 \end{cases}$. Find A and B so that $f(x)$ is continuous at all points. Sketch a graph of $y = f(x)$ for $-3 \le x \le 3$. For which values of x is $f(x)$ *not* differentiable?

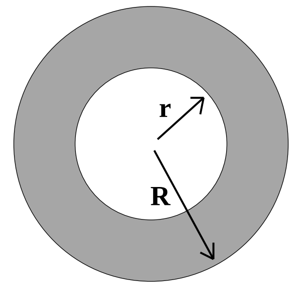
Ans.: $A = 7/3$, $B = -4/3$. f is not differentiable at -2 and 1.

4. In the graph of $y = f(x)$ to the right, identify with **m** any point which is a relative minimum; **M** any point which is a relative maximum; **C** any point which is a critical point; **I** any point which is an inflection point; **NC** any point at which $f(x)$ is *not* continuous; and **ND** any point at which $f(x)$ is *not* differentiable. Some points may have more than one label.



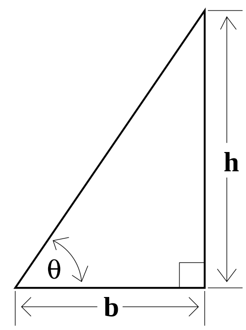
Related rates

1. Two circles have the same center. The inner circle has radius r which is increasing at the rate of 3 inches per second. The outer circle has radius R which is increasing at the rate of 2 inches per second. Suppose A is the area of the region *between* the circles. At a certain time, r is 7 inches and R is 10 inches. What is A at that time? How fast is A changing at that time? Is A increasing or decreasing at that time?



Ans.: $A = 51\pi$ square inches and A is decreasing at a rate of 2π square inches per second.

2. Suppose that the right triangle with height h and base b as shown here *always* has an area of 6 square inches. At a certain time, the length of b is 3 inches and it is increasing at the rate of .04 inches per minute. What is the length of h at that time? How fast is the length of h changing at that time? Is it increasing or decreasing?



Ans.: $h = 4$ and h is decreasing at the rate of $-0.16/3 = -0.05333\ldots$ inches per minute.

Linear approximation

1. Suppose you know that $f(5) = 7$, $f'(5) = 2$, $g(7) = 3$, and $g'(7) = 8$. If $F(x) = g(f(x))$, compute $F(5)$ and $F'(5)$. Use linear approximation or differentials to get an approximate value of $F(5.02)$.

Ans.: $F(5) = 3$, $F'(5) = 16$, $F(5.02)$ is approximately 3.32.

2. Suppose $f(x) = \tan(x^2)$. Use linear approximation or differentials to find an approximate value of $f\left(\sqrt{\frac{\pi}{4}} - .03\right)$.

Ans.: $1 - 0.06\sqrt{\pi} = 0.8936527\ldots$

Optimization

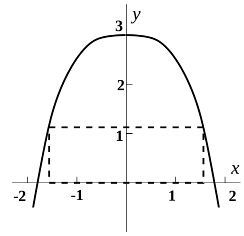
1. A box with an open top is to be made from a rectangular sheet of cardboard 5 inches by 8 inches by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. Use calculus to find the largest volume of the box. Be sure to explain briefly why your answer gives a maximum.

Ans.: The maximum volume is 18 cubic inches.

2. Find the largest and smallest values of $f(x) = 30x^3 - 90x^2 + 90x + 100$ when $2 \leq x \leq 3$. Justify your answer with calculus.

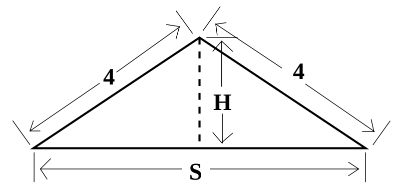
Ans.: The largest value is 370 and the smallest is 160.

3. A rectangle is inscribed as shown in the parabola $y = 3 - x^2$. Of all such rectangles, find the dimensions of the one whose area is a maximum. Explain briefly why your answer gives a maximum.



Ans.: The rectangle is a square with side 2.

4. A triangle has two equal sides each 4 inches long. What is the length of the third side if the triangle is to have maximum area? Be sure to explain why your answer is a maximum. The diagram may be useful. Use it to express an algebraic relationship between H and $\frac{1}{2}S$ and then compute the area of the triangle.



Ans.: The third side has length $4\sqrt{2}$

Intermediate Value Theorem & Mean Value Theorem

1. Suppose $f(x) = \frac{1}{101}x^{101} + \frac{5}{37}x^{37} + \frac{8}{5}x^5 + 46x + 8$. What is $f'(x)$? For which x is $f'(x)$ positive? For which x is $f'(x)$ negative? Describe briefly how calculus can verify that $f(77)$ is bigger than $f(33)$. Do *not* verify this by direct computation of function values.

Ans.: $f'(x)$ is positive for all x. Thus f is an increasing function, so $f(77)$ must be bigger than $f(33)$.

2. Suppose $M(x) = x^5 - 7x + 4$. Compute $M(2)$ and $M(-2)$ and explain briefly why the results allow you to conclude that $M(x) = 0$ has at least one root.

Ans.: $M(-2)M(2) < 0$, so by Intermediate Value Theorm there is a solution to $M(x) = 0$ in $[-2, 2]$.

Curve sketching

1. Suppose $f(x) = 5x^3 - 3x^5$. Compute $f'(x)$ and $f''(x)$. Where are each of these functions equal to 0? Find all relative max & min values of $f(x)$, briefly explaining your answers. Find all inflection points of $f(x)$, briefly explaining your answers.

Ans.: There is a relative minimum at $x = -1$ and a relative maximum at $x = 1$. There are points of inflection at $x = -\sqrt{2}/2$, $x = 0$, and $x = \sqrt{2}/2$. At $x = 0$ there is also a horizontal tangent.

2. In this problem the function G is defined by the formula $G(x) = (2 - e^x)(1 - e^x)$. Computations with exp and log should be simplified as much as possible; approximations are not acceptable.

a) What is $\lim_{x\to-\infty} G(x)$? What is $\lim_{x\to+\infty} G(x)$? Ans.: 2 and ∞.

b) For which x is $G(x) = 0$? Ans.: For 0 and $\ln(2)$.

c) Compute $G'(x)$. For which x is $G'(x) = 0$? Ans.: For $x = \ln(3/2)$.

d) Compute $G''(x)$. For which x is $G''(x) = 0$? Ans.: For $x = \ln(3/4)$.

e) Sketch a graph of $y = G(x)$, using the information you have obtained in the previous parts of the problem. Indicate precisely any relative maxima or minima or points of inflection with exact coordinates, and any vertical or horizontal asymptotes with exact equations.

Ans.: The line $y = 2$ is a horizontal asymptote. There is a relative minimum at $x = \ln(3/2)$ and a point of inflection at $x = \ln(3/4)$.

Antiderivatives & initial value problems

1. Suppose $f''(x) = x + \dfrac{1}{x^2}$ and $f(1) = 1$ and $f'(1) = -2$. Find a formula for $f(x)$. Computations with exp and log should be simplified as much as possible; approximations are not acceptable.

Ans.: $f(x) = \dfrac{x^3}{6} - \ln(x) - \dfrac{3x}{2} + \dfrac{7}{3}$

2. Suppose $f''(x) = \cos x + \sin(2x)$ and $f(0) = 0$ and $f'(\pi) = 0$. Find a formula for $f(x)$. Computations with sine and cosine should be simplified as much as possible; approximations are not acceptable.

Ans.: $f(x) = 1 + \dfrac{x}{2} - \cos(x) - \dfrac{\sin(2x)}{4}$

3. Suppose $f''(x) = e^x - 1$ and $f(0) = 2$ and $f'(0) = -3$. Find a formula for $f(x)$. Computations with exp and log should be simplified as much as possible; approximations are not acceptable.

Ans.: $f(x) = 1 - 4x - \dfrac{x^2}{2} + e^x$

Riemann sums

1. Suppose $f(x) = 2x^2 - 1$. Compute the Riemann sum for $f(x)$ on the interval $[-1, 5]$ with partition $\{-1, 2, 4, 5\}$ using the left-hand endpoints as sample points.

Ans.: 48

2. Suppose $f(x) = \cos x$. Compute the Riemann sum for $f(x)$ on the interval $[0, \frac{3}{2}\pi]$ obtained by partitioning the interval into 6 equal subintervals and using the right-hand endpoints as sample points.

Ans.: $-\dfrac{\pi(2+\sqrt{2})}{8}$

Definite integral

1. Suppose P and Q are constants, and $f(x) = P\sin(7x) + Qx\cos(7x)$. Find specific values of P and Q so that $f'(x) = x\sin(7x)$. Use your answer to evaluate $\int_9^{\pi/7} x\sin(7x)\,dx$.

Ans.: $P = \frac{1}{49}$, $Q = \frac{-1}{7}$, the integral is $\frac{\pi}{49} - \frac{\sin(63)}{49} + \frac{9\cos(63)}{7}$

2. Suppose $f(x) = xe^x - e^x$. Compute $f'(x)$, and use your answer to evaluate $\int_0^1 xe^x\,dx$ exactly.

Ans.: The value of the integral is 1.

Area

1. Sketch the region in the plane bounded by $y = 4 - x^2$ and the x-axis. Find the area of this region.

Ans.: The area is 32/3.

2. Sketch the region in the plane bounded above by $y = 4 - x^4$ and below by $y = 3$. Find the area of this region.

Ans.: The area is 4/3.

3. Sketch the region in the plane bounded by the x-axis, the line $x = 2$, and the curve $y = \frac{1}{9}x^5$. Find the area of this region.

Ans.: The area is 32/27.

Fundamental Theorem of Calculus

1. Find $\displaystyle\int_1^4 \left(2x - 5\sqrt{x}\right) dx$. Ans.: $-25/3$

2. Find $\displaystyle\int_1^4 \frac{1-\sqrt{x}}{x} dx$. Ans.: $2\ln(2) - 2$

3. If $f(x) = \displaystyle\int_{-42}^{x} \frac{\sin(t^2)}{1+t^4} dt$, compute $f(-42)$, $f'(0)$, and $f'(\sqrt{\pi})$ exactly.

Ans.: 0, 0, 0

4. Find $\displaystyle\int_1^2 \left(3\sqrt{x} - \frac{1}{x^4}\right) dx$. Ans.: $4\sqrt{2} - \frac{55}{24}$

Substitution

1. Find $\displaystyle\int (3x-6)e^{3x^2-12x} dx$. Ans.: $\frac{1}{2}e^{3x^2-12x} + C$

2. Find $\displaystyle\int_0^{\sqrt{\ln 7}} xe^{(x^2)} dx$. Ans.: 3