Partial Solutions to Review Problems for Exam 1

(1) The first inequality gives us the set which is the union of the intervals $(-\infty, 1)$ and $(3, \infty)$. The second inequality gives us the set which is just the interval $[2, 4]$. The intersection of these two sets gives us the answer to the given question. This answer is the interval $(3, 4]$.

(2)(a) We have to show that the inequalities $x_1 \le x_2 \le -5$ imply $f(x_1) \le f(x_2)$. The assumption $x_1 \le x_2 \le -5$ gives us $5 \le -x_2 \le -x_1$. Since $f(x)$ is odd and increasing on $[5, \infty)$, we conclude $-f(x_2) = f(-x_2) \le f(-x_1) = -f(x_1)$. Therefore, $f(x_1) \le f(x_2)$.

(2)(b) We have to show that the inequalities $x_1 \le x_2 \le -5$ imply $f(x_1) \ge f(x_2)$. The assumption $x_1 \le x_2 \le -5$ gives us $5 \le -x_2 \le -x_1$. Since $f(x)$ is even and increasing on $[5, \infty)$, we conclude $f(x_2) = f(-x_2) \le f(-x_1) = f(x_1)$. Therefore, $f(x_1) \ge f(x_2)$.

(3) To begin with, $2x^2 - 8x - 10 = 2(x^2 - 4x - 5) = 2\big((x-2)^2 - 9\big)$. The minimum of $2x^2 - 8x - 10$ occurs at the x which minimizes $(x-2)^2 - 9$. This particular x is the x that makes $(x-2)^2$ equal to 0. That x is 2. In other words, the minimum of $2x^2 - 8x - 10$ occurs at $x = 2$, and that minimum is -18. The equation $0 = 2x^2 - 8x - 10 = 2((x-2)^2 - 9)$ leads to $x - 2 = \pm 3$, $x = -1$ and $x = 5$.

(4) There are many choices for $f(x)$ and $g(x)$. The choices $f(x) = x^2$ and $g(x) = x + 1$ will work. Indeed, $(f \circ g)(x) = f(g(x)) = (x+1)^2$ does not equal $(g \circ f)(x) = g(f(x)) = x^2 + 1$ for general values of x.

(5) The given equation is equivalent to $2\sin^2 x = 1 + \cos^2 x - \sin^2 x = 1 + 1 - \sin^2 x - \sin^2 x$, which is equivalent to $\sin^2 x = 1/2$. We need to find all x in $[0, 2\pi]$ with the property $\sin x = \pm\sqrt{1/2} = \pm\sqrt{2}/2$. These x are $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$.

(6) The periodicity of sin gives $\sin^{-1}(\sin(9\pi/4)) = \sin^{-1}(\sin(\pi/4)) = \pi/4$. The fact $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ (see page 39 in the textbook) leads to $\sec(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. If we modify the argument in the text that led to the second proof of $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ on page 39, then we get the following: If $\theta = \tan^{-1} x$ then $\sec(\tan^{-1} x) = \sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + x^2}$, where we took the positive square root because $\theta = \tan^{-1} x$ lies in $(-\pi/2, \pi/2)$ and $\sec\theta$ is positive in this interval. Now we use the fact $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$ to conclude $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

(7) The given identity is equivalent to $\ln\left(\frac{x^2+7}{x^2+1}\right) = \ln\left(2^2\right)$. Since the ln function is one-to-one, we conclude $\frac{x^2+7}{x^2+1} = 2^2 = 4$. Consequently, $x^2 = 1$ and $x = \pm 1$.

(8) The average velocity is $\frac{1}{3-1}\left(\frac{3}{1+3^2} - \frac{1}{1+1^2}\right) = -0.1$ feet per second.

(9) We will use the letters (a) - (m) to label these 13 limit problems.

(a) If we take the reciprocal of $\lim_{u\to 0} \frac{\sin u}{u} = 1$ then the limit laws give us $\lim_{u\to 0} \frac{u}{\sin u} = 1$. If $u = 7x$ then $x \to 0$ is the same as $u \to 0$. Consequently,

$$\lim_{x\to 0} \frac{x}{\sin(7x)} = \frac{1}{7} \lim_{x\to 0} \frac{7x}{\sin(7x)} = \frac{1}{7} \lim_{u\to 0} \frac{u}{\sin u} = \frac{1}{7} \cdot 1 = \frac{1}{7}.$$

(b) Imitating the method in (a), we set $u = 5x$, $v = 7x$ and note that $x \to 0$ is the same as $u \to 0$ and the same as $v \to 0$. Consequently,

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(7x)} = \lim_{x\to 0} \frac{5}{7} \cdot \frac{\sin(5x)}{5x} \cdot \frac{7x}{\sin(7x)} = \frac{5}{7} \cdot \lim_{u\to 0} \frac{\sin u}{u} \cdot \lim_{v\to 0} \frac{v}{\sin v} = \frac{5}{7} \cdot 1 \cdot 1 = \frac{5}{7}.$$

(c) $\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot \cos x = 1 \cdot 1 = 1.$

(d) We are dealing with nonzero values of x. The inequality $|\cos(x^{-3})| \leq 1$ leads to $|x| \cdot |\cos(x^{-3})| \leq |x| \cdot 1$, which is $|x\cos(x^{-3})| \leq |x|$. This last inequality can be rewritten

$$-|x| \leq x\cos(x^{-3}) \leq |x|.$$

Since $\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$, the Squeeze Theorem implies $\lim_{x\to 0} x\cos(x^{-3}) = 0$.

(e) When x approaches 5 from the right, we always have $x > 5$, which is the same as $x - 5 > 0$. The inequality $x - 5 > 0$ allows us to write $|x - 5| = x - 5$. Now we know $\lim_{x\to 5^+} \frac{x-5}{|x-5|} = \lim_{x\to 5^+} \frac{x-5}{x-5} = \lim_{x\to 5^+} 1 = 1.$

(f) When x approaches 5 from the left, we always have $x < 5$, which is the same as $x - 5 < 0$. The inequality $x - 5 < 0$ allows us to write $|x - 5| = -(x - 5)$. Now we know $\lim_{x\to 5^-} \frac{x-5}{|x-5|} = \lim_{x\to 5^-} \frac{x-5}{-(x-5)} = \lim_{x\to 5^-} -1 = -1.$

(g) The general fact $\lim_{x\to a^+} \frac{1}{x-a} = \infty$ leads to

$$\lim_{x\to 3^+} \frac{x^2-20}{x^2-9} = \lim_{x\to 3^+} \left(\frac{x^2-20}{x+3} \cdot \frac{1}{x-3}\right) = \lim_{x\to 3^+} \frac{x^2-20}{x+3} \cdot \lim_{x\to 3^+} \frac{1}{x-3} = -\frac{11}{6}\infty = -\infty.$$

(h) The general fact $\lim_{x\to a^-} \frac{1}{x-a} = -\infty$ leads to

$$\lim_{x\to 3^-} \frac{x^2-20}{x^2-9} = \lim_{x\to 3^-} \left(\frac{x^2-20}{x+3} \cdot \frac{1}{x-3}\right) = \lim_{x\to 3^-} \frac{x^2-20}{x+3} \cdot \lim_{x\to 3^-} \frac{1}{x-3} = -\frac{11}{6}(-\infty) = \infty.$$

(i) $\lim_{x\to 2} \frac{x^2+x-6}{x^2+2x-8} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x+4)} = \lim_{x\to 2} \frac{x+3}{x+4} = \frac{5}{6}.$

(j) $\lim_{x\to 2} \dfrac{x^3-2x^2+x-2}{x^3-x^2-x-2} = \lim_{x\to 2} \dfrac{(x-2)(x^2+1)}{(x-2)(x^2+x+1)} = \lim_{x\to 2} \dfrac{x^2+1}{x^2+x+1} = \dfrac{5}{7}.$

(k) A correct rationalization starts with

$$\lim_{x\to 3} \frac{4-\sqrt{5x+1}}{5-\sqrt{8x+1}} = \lim_{x\to 3} \frac{(4-\sqrt{5x+1})(4+\sqrt{5x+1})(5+\sqrt{8x+1})}{(5-\sqrt{8x+1})(4+\sqrt{5x+1})(5+\sqrt{8x+1})}.$$

Using the basic identity $(a-b)(a+b) = a^2-b^2$ twice, we get

$$\lim_{x\to 3} \frac{4-\sqrt{5x+1}}{5-\sqrt{8x+1}} = \lim_{x\to 3} \frac{(16-(5x+1))(5+\sqrt{8x+1})}{(25-(8x+1))(4+\sqrt{5x+1})} = \lim_{x\to 3} \frac{5(3-x)(5+\sqrt{8x+1})}{8(3-x)(4+\sqrt{5x+1})}$$
$$= \lim_{x\to 3} \frac{5(5+\sqrt{8x+1})}{8(4+\sqrt{5x+1})} = \frac{5(5+5)}{8(4+4)} = \frac{25}{32}.$$

(l) $\lim_{x\to 3} \dfrac{4-\sqrt{5x+1}}{6-2x} = \lim_{x\to 3} \dfrac{(4-\sqrt{5x+1})(4+\sqrt{5x+1})}{(6-2x)(4+\sqrt{5x+1})} = \lim_{x\to 3} \dfrac{(16-(5x+1))}{(6-2x)(4+\sqrt{5x+1})} =$
$\lim_{x\to 3} \dfrac{5(3-x)}{2(3-x)(4+\sqrt{5x+1})} = \lim_{x\to 3} \dfrac{5}{2(4+\sqrt{5x+1})} = \dfrac{5}{16}.$

(m) $\lim_{x\to 0} \dfrac{1-\sec x}{x^2} = \lim_{x\to 0} \dfrac{(1-\sec x)(1+\sec x)}{x^2(1+\sec x)} = \lim_{x\to 0} \dfrac{(1-\sec^2 x)}{x^2(1+\sec x)} = \lim_{x\to 0} \dfrac{(-\tan^2 x)}{x^2(1+\sec x)} =$
$-\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^2 \dfrac{1}{(\cos^2 x)(1+\sec x)} = -(1)^2\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}.$

(10) We must have $2c = f(1) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{8}{1+x^2} = 4$. Now we know $c = 2$. The identity $4 = 2c = f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(bx+c) = b+c = b+2$ gives $b = 2$. Finally, we get $a = -2$ from $a+2 = a+b = f(-1) = \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(bx+c) = -b+c = -2+2 = 0$. One should check that the identities $\lim_{x\to -1} f(x) = f(-1)$ and $\lim_{x\to 1} f(x) = f(1)$ do hold for this choice of a, b, c.

(11) Let us use the auxiliary function $f(x) = x - \cos x$. We verify that $f(x)$ is continuous on $[0, \pi/2]$ and that we have $f(0) < 0 < f(\pi/2)$. The Intermediate Value Theorem tells us that $f(c) = 0$ is true for some c in the interval $(0, \pi/2)$. This c is a solution of $x = \cos x$.

(12) For each $\varepsilon > 0$ we can choose $\delta = \varepsilon/3 > 0$ and verify that the condition $0 < |x-2| < \delta$ implies $|(3x+4) - 10| = |3x-6| = 3|x-2| < 3\delta = \varepsilon$.

(13) Using $f(x) = \frac{1}{x^2}$, $f(x+h) = \frac{1}{(x+h)^2}$ we get

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$
$$= \lim_{h\to 0} \frac{x^2-(x+h)^2}{h(x+h)^2x^2}$$
$$= \lim_{h\to 0} \frac{-2xh-h^2}{h(x+h)^2x^2}$$
$$= \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^2x^2} = -2x^{-3}.$$

(14) If t is the time when the outfielder caught the ball, then we must have $-16t^2+12t+4 = 6$. The two solutions are $t = 1/4$ seconds (when the ball is on its way up) and $t = 1/2$ seconds (when the ball is on its way down). I may have seen once a pitcher who caught a batted ball on its way up, but an outfielder expects to catch a fly ball on its way down. If the catch was routine then the answer to the first question is $t = 1/2$ seconds. The maximum height occurs when the derivative of the height function is zero. This corresponds to $t = 3/8$ seconds and the maximum height $-16(3/8)^2 + 12(3/8) + 4$ feet.

(15) We see $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+3) = 2+3 = 5 = 3+2 = \lim_{x\to 1^+}(3x+2) = \lim_{x\to 1^+} f(x)$. This implies $\lim_{x\to 1} f(x) = 5$. We also see $f(1) = 3+2 = 5$. Now we have $\lim_{x\to 1} f(x) = f(1)$, which proves continuity at 1, the only point where continuity could possibly be in doubt. Now we will show that $f'(1)$ does not exist. If $f'(1)$ did exist, then $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ would exist, and this would imply

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h},$$

which is the same as

$$\lim_{h\to 0^-} \frac{f(1+h)-5}{h} = \lim_{h\to 0^+} \frac{f(1+h)-5}{h}.$$

We get a contradiction because the computations

$$\lim_{h\to 0^-} \frac{f(1+h)-5}{h} = \lim_{h\to 0^-} \frac{(2(1+h)+3)-5}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$$

and

$$\lim_{h\to 0^+} \frac{f(1+h)-5}{h} = \lim_{h\to 0^+} \frac{(3(1+h)+2)-5}{h} = \lim_{h\to 0^+} \frac{3h}{h} = 3$$

tell us that the earlier assertion

$$\lim_{h\to 0^-} \frac{f(1+h)-5}{h} = \lim_{h\to 0^+} \frac{f(1+h)-5}{h}$$

implies the conclusion $2 = 3$, which is false. A quick way to visualize the nondifferentiability of this $f(x)$ is to note the geometry at 1 on the x-axis. The slope from the right is 3, but the slope from the left is 2. This produces a corner at 1.

(16) Since $f(x)$ is differentiable at 2, we conclude that $f(x)$ is continuous at 2. This implies $5 = \lim_{x\to 2^-}(2x+1) = \lim_{x\to 2^-} f(x) = f(2) = 4a+b$. This gives us the equation $5 = 4a+b$, which relates a and b. As in problem (15), we can visualize what is going on at the point 2 on the x-axis. The slope from the right is $2ax$ at $x = 2$, which is $4a$. The slope from the left is 2. To avoid a corner, we need $4a = 2$. What follows is a more rigorous analysis that also leads to $4a = 2$. The differentiability of $f(x)$ at 2 implies

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}.$$

The above, the fact $f(2) = 4a + b = 5$ and the definition of f give

$$\lim_{h\to 0^-} \frac{(2(2+h)+1) - 5}{h} = \lim_{h\to 0^+} \frac{(a(2+h)^2 + b) - 5}{h},$$

which simplifies to

$$\lim_{h\to 0^-} \frac{2h}{h} = \lim_{h\to 0^+} \frac{4ah + ah^2}{h}.$$

We used $4a + b = 5$ in this simplification. Further simplification leads to $\lim_{h\to 0^-} 2 = \lim_{h\to 0^+}(4a + ah)$, which is $2 = 4a$, as promised earlier. Solving the system $5 = 4a + b$ and $2 = 4a$, we get $b = 3$ and $a = 1/2$.

(17) The Product Rule, Quotient Rule, Chain Rule and other simpler rules give

$$\frac{d}{dx}\left[(x^3+x)^5(1+\cos x)^9\right] = 5(x^3+x)^4(3x^2+1)(1+\cos x)^9 + (x^3+x)^5 9(1+\cos x)^8(-\sin x)$$

$$\frac{d}{dx}\left[\frac{\tan x}{1+e^{4x}}\right] = \frac{(\sec^2 x)(1+e^{4x}) - 4e^{4x}\tan x}{(1+e^{4x})^2}$$

$$\frac{d}{dx}\left[\sin\left(\sqrt{x^4+x^2+3}\right)\right] = \frac{1}{2}(x^4+x^2+3)^{-1/2}(4x^3+2x)\cos\left(\sqrt{x^4+x^2+3}\right)$$

$$\frac{d}{dx}\left[\sec(e^x+\sqrt{x})\right] = (e^x + (1/2)x^{-1/2})\sec(e^x+\sqrt{x})\tan(e^x+\sqrt{x}).$$

(18) If $f(x) = (3 + x^{-3})^5$ then $f''(x) = 60x^{-5}(3 + x^{-3})^4 + 180x^{-8}(3 + x^{-3})^3$.
If $g(x) = \tan(7x)$ then $g''(x) = 98\sec^2(7x)\tan(7x)$.
If $h(x) = (e^x + \cos x)^{-1/2}$ then

$$h''(x) = \frac{3}{4}(e^x + \cos x)^{-5/2}(e^x - \sin x)^2 - \frac{1}{2}(e^x + \cos x)^{-3/2}(e^x - \cos x).$$

If $k(x) = e^{x^2+4x+3}$ then $k''(x) = \big((2x+4)^2 + 2\big)e^{x^2+4x+3}$.

(19) If $f(x) = \cos(2x)$ then $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, $f^{(3)}(x) = 8\sin(2x)$, $f^{(4)}(x) = 16\cos(2x)$.

(20) The identity $0 = f''(x) = (4x^2 - 2)e^{-x^2}$ gives the solutions $x = \pm 2^{-1/2}$. The identity $0 = g''(x) = \frac{6x^2-2}{(1+x^2)^3}$ gives the solutions $x = \pm 3^{-1/2}$.