Solutions to the additional review problems

(1) We find

$$\int_1^4 f(x)\,dx = \int_1^{10} f(x)\,dx - \int_4^{10} f(x)\,dx = 2 - 7 = -5\ .$$

Using this, we get

$$\int_4^8 f(x)\,dx = \int_1^8 f(x)\,dx - \int_1^4 f(x)\,dx = 14 - (-5) = 19\ .$$

(2) We know $\frac{d}{dx}\int_{x^3}^0 e^{t^2}\,dt = \frac{d}{dx}\left(-\int_0^{x^3} e^{t^2}\,dt\right) = -\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dt$. Therefore, the problem reduces to the evaluation of $\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dt$. If $F(x) = \int_0^x e^{t^2}\,dt$ then the fact $F'(x) = e^{x^2}$ (second version of the Fundamental Theorem of Calculus) implies

$$\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dt = \frac{d}{dx}F(x^3) = 3x^2F'(x^3) = 3x^2e^{(x^3)^2} = 3x^2e^{x^6}\ .$$

This gives the answer

$$\frac{d}{dx}\int_{x^3}^0 e^{t^2}\,dt = -3x^2e^{x^6}$$

to the original question.

(3) First, we solve the easier problem $\lim_{x\to\infty} \ln\left(\left(1+\frac{3}{x}\right)^x\right)$. Then we do problem (3) using the solution to the easier problem.

$$\lim_{x\to\infty} \ln\left(\left(1+\frac{3}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+3/x)}{1/x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(1+3/x)}{\frac{d}{dx}(1/x)}$$
$$= \lim_{x\to\infty}\frac{\left(\frac{-3/x^2}{1+3/x}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{3}{1+3/x} = 3\ .$$

When we apply the exponential function to the above, we get the solution

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = e^3$$

to problem (3).

(4) At time t, the size of the population is Ae^{tk}, for some constants A and k. Since it quadruples in size between time 0 and time 7, we see $Ae^{7k} = 4Ae^{0k}$, which simplifies to $e^{7k} = 4$, $7k = \ln 4$, $k = (\ln 4)/7$. If T is the time it takes for the population to triple, then $Ae^{Tk} = 3Ae^{0k}$, which simplifies to $e^{Tk} = 3$, $Tk = \ln 3$, $T = (\ln 3)/k$. Plugging $k = (\ln 4)/7$ into the last identity of the previous sentence, we get $T = \frac{7\ln 3}{\ln 4}$. It takes $\dfrac{7\ln 3}{\ln 4}$ days for the population to triple in size.

(5) The substitution $u = x^4$, $du = 4x^3\,dx$, $(1/4)du = x^3\,dx$ gives

$$\int \frac{x^3\,dx}{\sqrt{1-x^8}} = \int \frac{x^3\,dx}{\sqrt{1-(x^4)^2}} = \int \frac{(1/4)du}{\sqrt{1-u^2}} = (1/4)\sin^{-1} u + C = (1/4)\sin^{-1}(x^4) + C\ .$$

(6) The substitution $u = x^4$, $du = 4x^3\,dx$, $(1/4)du = x^3\,dx$ gives

$$\int \frac{x^3\,dx}{1+x^8} = \int \frac{x^3\,dx}{1+(x^4)^2} = \int \frac{(1/4)du}{1+u^2} = (1/4)\tan^{-1} u + C = (1/4)\tan^{-1}(x^4) + C\ .$$

(7) The first derivative of $x^2 \ln x$ is $2x\ln x + x^2(1/x) = 2x\ln x + x$. The second derivative is $\dfrac{d}{dx}\big(2x\ln x + x\big) = 2\ln x + 2x/x + 1 = 2\ln x + 3$. The second derivative is negative when $\ln x < -3/2$, and the second derivative is positive when $\ln x > -3/2$. When $x > 0$, the inequalities $\ln x < -3/2$ and $\ln x > -3/2$ are equivalent to $x < e^{-3/2}$ and $x > e^{-3/2}$, respectively. Consequently, the function is concave down on $(0, e^{-3/2})$ and it is concave up on $(e^{-3/2}, \infty)$.

(8) The function $|x-1|$ is defined in pieces:

$$|x-1| = \begin{cases} -(x-1) & \text{if } x \le 1 \\ x-1 & \text{if } x \ge 1 \end{cases}$$

Therefore,

$$\begin{aligned}\int_{-2}^{5} |x-1|\,dx &= \int_{-2}^{1} |x-1|\,dx + \int_{1}^{5} |x-1|\,dx = \int_{-2}^{1} -(x-1)\,dx + \int_{1}^{5} x-1\,dx \\ &= \int_{-2}^{1} 1-x\,dx + \int_{1}^{5} x-1\,dx = \left(x - \frac{x^2}{2}\right)\Big|_{-2}^{1} + \left(\frac{x^2}{2} - x\right)\Big|_{1}^{5}\ .\end{aligned}$$

(9) We wish to minimize the distance between $(10,0)$ and the curve $x = 2y^2$ for $x \geq 0$. In other words, we wish to minimize

$$\text{(I)} \qquad \sqrt{(x-10)^2 + (y-0)^2} = \sqrt{(x-10)^2 + y^2} = \sqrt{(x-10)^2 + x/2}$$

over the interval $x \geq 0$. The nonnegative x that minimize (I) are the x that solve the following problem:

$$\text{(II)} \qquad \text{Minimize } (x-10)^2 + x/2 \text{ over } x \geq 0 \ .$$

When we look for critical points in (II), we find

$$0 = \frac{d}{dx}\big((x-10)^2 + x/2\big) = 2(x-10) + 1/2 \ , \quad 0 = 2x - 39/2 \ , \quad x = 39/4 \ .$$

Since the function $(x-10)^2+x/2$ has positive second derivative, we conclude that $x = 39/4$ solves the minimizing problem (II). This corresponds to the points $(39/4, \pm\sqrt{39/8})$ on the curve $x = 2y^2$.

(10) The first derivative is $\dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$. Therefore, the second derivative is

$$\frac{d}{dx}(1-x^2)^{-1/2} = (-1/2)(1-x^2)^{-3/2}(-2x) = \frac{x}{(1-x^2)^{3/2}} \ .$$

(11) The two curves intersect when $y^2 = y + 2$. This gives the two values $y = -1$ and $y = 2$. Since $(y+2) - y^2 = -(y+1)(y-2)$, we conclude $(y+2) - y^2 \geq 0$ for $-1 \leq y \leq 2$. This means $y^2 \leq y + 2$ for $-1 \leq y \leq 2$. The area is

$$\int_{-1}^{2} (y+2) - y^2 \, dy = \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\Bigg|_{-1}^{2} \ .$$

(12) We will use the formula $\displaystyle\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$. We get

$$\sum_{i=1000}^{2000} i = \sum_{i=1}^{2000} i - \sum_{i=1}^{999} i = \frac{2000 \cdot 2001}{2} - \frac{999 \cdot 1000}{2} \ .$$