Solutions to Math 152 Review Problems for Exam 1

(1) If $A(x)$ is the area of the rectangle formed when the solid is sliced at x perpendicular to the x-axis, then $A(x) = |x|(2\sqrt{1-x^2})$, because the height of the rectangle is $|x|$ and the base of the rectangle has length $2\sqrt{1-x^2}$. Therefore, the volume of the solid is

$$\begin{aligned}\int_{-1}^{1} |x|(2\sqrt{1-x^2})\,dx &= \int_{-1}^{0} |x|(2\sqrt{1-x^2})\,dx + \int_{0}^{1} |x|(2\sqrt{1-x^2})\,dx \\ &= \int_{-1}^{0} (-x)(2\sqrt{1-x^2})\,dx + \int_{0}^{1} x(2\sqrt{1-x^2})\,dx.\end{aligned}$$

If we use the substitution $u = 1 - x^2$ then we see that the integral over $[-1, 0]$ is equal to $2/3$, and that the integral over $[0, 1]$ is also equal to $2/3$. Therefore, the volume of the solid is $2/3 + 2/3 = 4/3$.

(2) The average of $f(x) = \sqrt{1-x^2}$ over $[-1, 1]$ is $\frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$. We used the following fact: $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ is half of the area of a circle of radius 1. Now we have to solve $\sqrt{1-c^2} = f(c) = \frac{\pi}{4}$. The solutions are $c = \pm\sqrt{1-\pi^2/16}$. The existence of at least one c is guaranteed by the Mean Value Theorem for Integrals.

(3)(a) This type of cone is obtained when the region bounded by $y = 0$, $x = H$, $y = (R/H)x$ is rotated about the x-axis. The volume is $\pi\int_0^H ((R/H)x)^2\,dx = \pi R^2 H/3$.

(3)(b) This type of cone is obtained when the region bounded by $x = 0$, $y = H$, $y = (H/R)x$ is rotated about the y-axis. The volume is $\int_0^R 2\pi x(H - (H/R)x)\,dx = \pi R^2 H/3$.

(4)(a) The equation $y = 2(x-3)$ is equivalent to $x = 3 + y/2$. The method of washers gives a volume of

$$\pi\int_0^2 (3 + y/2 - 1)^2 - (3-1)^2\,dy = \pi\int_0^2 2y + y^2/4\,dy = \pi(4 + 2/3) = 14\pi/3.$$

(4)(b) The method of shells gives a volume of

$$\int_3^4 2\pi(x-1)(2 - 2(x-3))\,dx = 4\pi\int_3^4 -x^2 + 5x - 4\,dx = 4\pi(7/6) = 14\pi/3.$$

(5)(a) $u = \sin x$, $\displaystyle\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

(5)(b) $u = \cos x$, $\displaystyle\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C,$ where the last expression equals $\ln|\sec x| + C$.

(5)(c) $u = \sec x$, $\displaystyle\int \tan x\,dx = \int \frac{\tan x \sec x\,dx}{\sec x} = \int \frac{du}{u} = \ln|\sec x| + C.$

(5)(d) $u = \cosh x$, $\displaystyle\int \tanh x\,dx = \int \frac{\sinh x\,dx}{\cosh x} = \int \frac{du}{u} = \ln|u| + C = \ln|\cosh x| + C,$ which equals $\ln(\cosh x) + C$ because $\cosh x$ is always positive for real values of x.

(5)(e) $u = \sinh x$, $\displaystyle\int \coth x\,dx = \int \frac{\cosh x\,dx}{\sinh x} = \int \frac{du}{u} = \ln|u| + C = \ln|\sinh x| + C.$

(6)(A) When $u = \csc x + \cot x$ we get

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x}\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx \\ &= -\int \frac{du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C.\end{aligned}$$

(6)(B) When $u = \csc x - \cot x$ we get

$$\begin{aligned}\int \csc x\,dx &= \int \csc x \cdot \frac{\csc x - \cot x}{\csc x - \cot x}\,dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\,dx \\ &= \int \frac{du}{u} = \ln|u| + C = \ln|\csc x - \cot x| + C.\end{aligned}$$

(6)(C) The computation below uses $\csc^2 x - \cot^2 x = 1$ at the end:

$$\begin{aligned}-\ln|\csc x + \cot x| &= \ln\left|\frac{1}{\csc x + \cot x}\right| = \ln\left|\frac{\csc x - \cot x}{(\csc x + \cot x)(\csc x - \cot x)}\right| \\ &= \ln\left|\frac{\csc x - \cot x}{\csc^2 x - \cot^2 x}\right| = \ln|\csc x - \cot x|.\end{aligned}$$

(7) For the first integral, we do an integration by parts and get

$$\begin{aligned}\int \sin^2 x\,dx &= \int \sin x \sin x\,dx \\ &= (-\cos x)(\sin x) - \int (-\cos x)(\cos x)\,dx \\ &= -\cos x \sin x + \int \cos^2 x\,dx \\ &= -\cos x \sin x + \int (1 - \sin^2 x)\,dx \\ &= -\cos x \sin x + x - \int \sin^2 x\,dx.\end{aligned}$$

After adding $\int \sin^2 x\,dx$ to both sides, we discover

$$2\int \sin^2 x\,dx = -\cos x \sin x + x + C,$$

and this is equivalent to

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \cos x \sin x) + C.$$

The second integral is very similar. Integration by parts produces

$$\begin{aligned}
\int \cos^2 x\,dx &= \int \cos x \cos x\,dx \\
&= (\sin x)(\cos x) - \int (\sin x)(-\sin x)\,dx \\
&= \sin x \cos x + \int \sin^2 x\,dx \\
&= \sin x \cos x + \int (1 - \cos^2 x)\,dx \\
&= \sin x \cos x + x - \int \cos^2 x\,dx.
\end{aligned}$$

We add $\int \cos^2 x\,dx$ to both sides and get

$$2\int \cos^2 x\,dx = \sin x \cos x + x + C,$$

and this is equivalent to

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C.$$

The method in this problem can be used to derive the reduction formulas for $\int \cos^n x\,dx$ and $\int \sin^n x\,dx$. In order to get the cos reduction formula, we proceed as follows:

$$\begin{aligned}
\int \cos^n x\,dx &= \int \cos x \cos^{n-1} x\,dx \\
&= (\sin x)(\cos^{n-1} x) - \int (\sin x)\big((n-1)\cos^{n-2} x(-\sin x)\big)\,dx \\
&= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\
&= \sin x \cos^{n-1} x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x\,dx \\
&= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.
\end{aligned}$$

After adding $(n-1)\int \cos^n x\,dx$ to both sides, we get

$$n\int \cos^n x\,dx = \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx,$$

and this is equivalent to

$$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

(8) If we use the substitution $u = \tan x$ then we find

$$\int \tan x\sec^4 x\,dx = \int \tan x\sec^2 x\sec^2 x\,dx = \int \tan x(1+\tan^2 x)\sec^2 x\,dx$$
$$= \int u(1+u^2)\,du = \int u + u^3\,du = \frac{u^2}{2} + \frac{u^4}{4} + C = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C.$$

On the other hand, we can use the substitution $u = \sec x$ and obtain

$$\int \tan x\sec^4 x\,dx = \int \sec^3 x(\tan x\sec x)\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{\sec^4 x}{4} + C.$$

To see that these two answers are really the same, note

$$\frac{\sec^4 x}{4} + C = \frac{(\sec^2 x)^2}{4} + C = \frac{(1+\tan^2 x)^2}{4} + C = \frac{1+2\tan^2 x+\tan^4 x}{4} + C$$
$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + \left(\frac{1}{4} + C\right) = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

because $\frac{1}{4} + C$ is an arbitrary constant.

(9) We use integration by parts and long division (applied to $\frac{x^2}{1+x^2}$) to conclude

$$\int x\tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C.$$

(10) There are three integrations by parts:

$$\int (\ln x)^3\,dx = x(\ln x)^3 - \int x\cdot\frac{3(\ln x)^2}{x}\,dx$$
$$= x(\ln x)^3 - 3\int (\ln x)^2\,dx$$
$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - \int x\cdot\frac{2\ln x}{x}\,dx\right]$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\int \ln x\,dx$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x\ln x - \int \frac{x}{x}\,dx\right]$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C.$$

Of course, we could have derived a reduction formula for $\int (\ln x)^n \, dx$ and used it three times.

(11) The first integration by parts gets us as far as

$$\begin{aligned}\int \cos(ax)\cos(bx)\,dx &= \big(\frac{1}{a}\sin(ax)\big)\cos(bx) - \int \frac{1}{a}\sin(ax)\big(-b\sin(bx)\big)\,dx \\ &= \frac{1}{a}\sin(ax)\cos(bx) + \frac{b}{a}\int \sin(ax)\sin(bx)\,dx.\end{aligned}$$

Now we focus on $\int \sin(ax)\sin(bx)\,dx$ and do another integration by parts:

$$\begin{aligned}\int \sin(ax)\sin(bx)\,dx &= \big(-\frac{1}{a}\cos(ax)\big)\sin(bx) - \int \big(-\frac{1}{a}\cos(ax)\big)b\cos(bx)\,dx \\ &= -\frac{1}{a}\cos(ax)\sin(bx) + \frac{b}{a}\int \cos(ax)\cos(bx)\,dx.\end{aligned}$$

When this equation is substituted into the previous equation, we get

$$\begin{aligned}&\int \cos(ax)\cos(bx)\,dx \\ &= \frac{1}{a}\sin(ax)\cos(bx) + \frac{b}{a}\left[-\frac{1}{a}\cos(ax)\sin(bx) + \frac{b}{a}\int \cos(ax)\cos(bx)\,dx\right],\end{aligned}$$

which can be rewritten as

$$\left(1 - \frac{b^2}{a^2}\right)\int \cos(ax)\cos(bx)\,dx = \frac{1}{a}\sin(ax)\cos(bx) - \frac{b}{a^2}\cos(ax)\sin(bx) + C.$$

This is equivalent to

$$\int \cos(ax)\cos(bx)\,dx = \frac{a\sin(ax)\cos(bx) - b\cos(ax)\sin(bx)}{a^2 - b^2} + C.$$

One can show that this answer is equivalent to equation 25 on page 410 of the textbook.

(12) Integration by parts and the trigonometric identity $1 + \tan^2 x = \sec^2 x$ allow us to write

$$\begin{aligned}\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx &= \tan x \sec x - \int \tan x(\tan x \sec x)\,dx \\ &= \tan x \sec x - \int \tan^2 x \sec x\,dx \\ &= \tan x \sec x - \int (\sec^2 x - 1)\sec x\,dx \\ &= \tan x \sec x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \tan x \sec x - \int \sec^3 x\,dx + \ln|\tan x + \sec x|.\end{aligned}$$

After adding $\int \sec^3 x\,dx$ to both sides, we get

$$2\int \sec^3 x\,dx = \tan x \sec x + \ln|\tan x + \sec x| + C.$$

After dividing by 2, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln|\tan x + \sec x| + C.$$

The method in this problem can be used to derive the reduction formulas for $\int \sec^m x\,dx$ and $\int \csc^m x\,dx$. In order to get the sec reduction formula, we proceed as follows:

$$\begin{aligned}
\int \sec^m x\,dx &= \int \sec^2 x \sec^{m-2} x\,dx \\
&= (\tan x)(\sec^{m-2} x) - \int (\tan x)\big((m-2)\sec^{m-3} x \tan x \sec x\big)\,dx \\
&= \tan x \sec^{m-2} x - (m-2)\int \tan^2 x \sec^{m-2} x\,dx \\
&= \tan x \sec^{m-2} x - (m-2)\int (\sec^2 x - 1)\sec^{m-2} x\,dx \\
&= \tan x \sec^{m-2} x - (m-2)\int \sec^m x\,dx + (m-2)\int \sec^{m-2} x\,dx.
\end{aligned}$$

After adding $(m-2)\int \sec^m x\,dx$ to both sides, we get

$$(m-1)\int \sec^m x\,dx = \tan x \sec^{m-2} x + (m-2)\int \sec^{m-2} x\,dx,$$

and this is equivalent to

$$\int \sec^m x\,dx = \frac{1}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx.$$

(13) We use the substitution $x = 3\tan\theta$ and the result of problem (12):

$$\begin{aligned}
\int \sqrt{9+x^2}\,dx &= \int (3\sec\theta)(3\sec^2\theta)\,d\theta = 9\int \sec^3\theta\,d\theta \\
&= \frac{9}{2}\tan\theta\sec\theta + \frac{9}{2}\ln|\tan\theta + \sec\theta| + C \\
&= \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9+x^2}}{3} + \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{9+x^2}}{3}\right| + C \\
&= \frac{x}{2}\sqrt{9+x^2} + \frac{9}{2}\ln(x + \sqrt{9+x^2}) + C,
\end{aligned}$$

where we used

$$\frac{9}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{9+x^2}}{3}\right|+C=\frac{9}{2}\ln|x+\sqrt{9+x^2}|-\frac{9}{2}\ln 3+C,$$

the fact that $-\frac{9}{2}\ln 3+C$ is an arbitrary constant C, and the positivity of $x+\sqrt{9+x^2}$ for any x.

(14) Here the substitution $x=3\sin\theta$ is appropriate and we use problem (7):

$$\begin{aligned}\int\sqrt{9-x^2}\,dx&=\int 3\cos\theta\cdot 3\cos\theta\,d\theta=9\int\cos^2\theta\,d\theta\\&=\frac{9}{2}(\theta+\sin\theta\cos\theta)+C=\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right)+\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)+C.\end{aligned}$$

(15) Here the substitution $x=\tan\theta$ is appropriate and we also use problem (7). The trick $\sin\theta\cos\theta=\frac{\tan\theta}{\sec^2\theta}$ is helpful at the end when we rewrite everything in terms of x.

$$\begin{aligned}\int\frac{dx}{(1+x^2)^2}&=\int\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2}=\int\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}=\int\frac{d\theta}{\sec^2\theta}=\int\cos^2\theta\,d\theta\\&=\frac{1}{2}(\theta+\sin\theta\cos\theta)+C=\frac{1}{2}\left(\theta+\frac{\tan\theta}{\sec^2\theta}\right)+C\\&=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{1+x^2}\right)+C.\end{aligned}$$

(16) The correct partial fractions setup is

$$\int\frac{3x^2-3x-2}{(x^2-1)(x-1)}\,dx=\int\frac{3x^2-3x-2}{(x+1)(x-1)^2}\,dx=\int\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\,dx.$$

Now we have to solve for A, B, C in the equation

$$\frac{3x^2-3x-2}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}.$$

Since this can be rewritten as

$$\frac{3x^2-3x-2}{(x+1)(x-1)^2}=\frac{A(x-1)^2+B(x-1)(x+1)+C(x+1)}{(x+1)(x-1)^2},$$

we have to solve for A, B, C in the equation

$$3x^2-3x-2=A(x-1)^2+B(x-1)(x+1)+C(x+1).$$

If we plug in $x = 1$ and $x = -1$ into this last equation, then we get $C = -1$ and $A = 1$, respectively. If we compare the coefficients of x^2 in this same last equation, then we get $3 = A + B$. Now we conclude $B = 2$. Finally,

$$\begin{aligned} \int \frac{3x^2 - 3x - 2}{(x^2-1)(x-1)}\,dx &= \int \frac{1}{x+1} + \frac{2}{x-1} + \frac{-1}{(x-1)^2}\,dx \\ &= \ln|x+1| + 2\ln|x-1| + \frac{1}{x-1} + C. \end{aligned}$$

(17) The correct partial fractions setup is

$$\int \frac{x^2+3x}{(x^2+1)(x+1)}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{C}{x+1}\,dx.$$

Now we have to solve for A, B, C in the equation

$$\frac{x^2+3x}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} = \frac{(Ax+B)(x+1) + C(x^2+1)}{(x^2+1)(x+1)}.$$

This is the same as solving for A, B, C in the equation

$$(*) \qquad x^2 + 3x = (Ax+B)(x+1) + C(x^2+1) = (A+C)x^2 + (B+A)x + (B+C).$$

Equating coefficients, we get

$$(**) \qquad A + C = 1 \quad , \quad B + A = 3 \quad , \quad B + C = 0.$$

There is a short cut for solving these three equations in three unknowns: If we go back to the equation $x^2 + 3x = (Ax+B)(x+1) + C(x^2+1)$ in $(*)$ and plug in $x = -1$, then we obtain $-2 = 2C$, hence $C = -1$. Plugging $C = -1$ into part $A + C = 1$ of $(**)$ we get $A = 2$. Plugging $A = 2$ into $B + A = 3$ of $(**)$ we get $B = 1$. Our results are consistent with $B + C = 0$ of $(**)$. This is a way of checking our arithmetic. Plugging these A, B, C into our partial fractions setup, we get

$$\int \frac{x^2+3x}{(x^2+1)(x+1)}\,dx = \int \frac{2x+1}{x^2+1} + \frac{-1}{x+1}\,dx = \ln(x^2+1) + \tan^{-1} x - \ln|x+1| + C.$$

(18) We know $0 < \frac{1}{x} < \frac{e^x}{x}$ for $0 < x \leq 1$ and we know that $\int_0^1 \frac{1}{x}\,dx$ diverges. All this implies that $\int_0^1 \frac{e^x}{x}\,dx$ diverges.

Turning to the other improper integral, we obtain $\int_0^\infty xe^{-x^4}\,dx = \frac{1}{2}\int_0^\infty e^{-u^2}\,du$ when we use the substitution $u = x^2$. If we can show that $\int_0^\infty e^{-u^2}\,du$ converges then we will be able to conclude that $\int_0^\infty xe^{-x^4}\,dx$ converges. It is clear that $\int_0^1 e^{-u^2}\,du$ is finite. If we can show that $\int_1^\infty e^{-u^2}\,du$ is finite then it will be clear that

$$\int_0^\infty e^{-u^2}\,du = \int_0^1 e^{-u^2}\,du + \int_1^\infty e^{-u^2}\,du$$

is finite, and hence convergent. If $u \geq 1$ then $0 < e^{-u^2} \leq e^{-u}$. In addition, $\int_1^\infty e^{-u}\,du = e^{-1}$. The last two sentences imply that $\int_1^\infty e^{-u^2}\,du$ is finite.

(19) We solve $\dfrac{1}{(2x+1)(3x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{3x+1}$. This is equivalent to $A(3x+1) + B(2x+1) = 1$. The substitution $x = -1/3$ gives $B = 3$. The substitution $x = -1/2$ gives $A = -2$. Now we get

$$\int \frac{dx}{(2x+1)(3x+1)} = \int \frac{3}{3x+1} - \frac{2}{2x+1}\,dx = \ln|3x+1| - \ln|2x+1| + C = \ln\left|\frac{3x+1}{2x+1}\right| + C.$$

The computation

$$\lim_{R\to\infty} \ln\left|\frac{3R+1}{2R+1}\right| = \ln\left|\frac{3}{2}\right| = \ln(3/2)$$

leads to $\displaystyle\int_4^\infty \frac{dx}{(2x+1)(3x+1)} = \lim_{R\to\infty} \ln\left|\frac{3R+1}{2R+1}\right| - \ln\left|\frac{3\cdot 4+1}{2\cdot 4+1}\right| = \ln(3/2) - \ln(13/9).$

(20) We know $0 \leq \dfrac{\cos^2 x}{x^3} \leq \dfrac{1}{x^3}$ for $x \geq 1$. Since $\int_1^\infty \dfrac{1}{x^3}\,dx$ converges, we conclude that $\int_1^\infty \dfrac{\cos^2 x}{x^3}\,dx$ converges.