Solutions to Review sheet for Exam 2 of Math 152

(1) We will use the Limit Comparison Test with the simpler series

$$\sum_{n=2}^{\infty} \sqrt{\frac{1}{n^2}}.$$

The limit of the ratios is

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n^2-n-1}{n^4+n^3+5n^2+7}}}{\sqrt{\frac{1}{n^2}}} = \lim_{n\to\infty} \sqrt{\frac{n^2-n-1}{n^4+n^3+5n^2+7}\cdot n^2} = \lim_{n\to\infty} \sqrt{\frac{n^4-n^3-n^2}{n^4+n^3+5n^2+7}} = 1,$$

which is a finite, positive number. If the simpler series $\sum_{n=2}^{\infty} \sqrt{\frac{1}{n^2}}$ diverges then the original series diverges. The simpler series is just $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges. Therefore, the original series diverges.

(2) This is an example where the Limit Comparison Test is used twice. Consider first the simpler series $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2+5}$. We do a Limit Comparison of this simpler series with the even simpler series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$. The limit of the ratios is

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n-1}{n^2+5}\cdot n = \lim_{n\to\infty} \frac{n^2-n}{n^2+5} = 1,$$

which is a finite, positive number. Since the series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we conclude that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2+5}$ also diverges. Now we do a Limit Comparison Test involving $\sum_{n=2}^{\infty} \frac{n-1}{n^2+5}$ (which we now know to be a divergent series) and the original series of the problem. The limit of the ratios is

$$\lim_{n\to\infty} \frac{\sin\left(\frac{n-1}{n^2+5}\right)}{\frac{n-1}{n^2+5}} = 1$$

because $\frac{n-1}{n^2+5}$ approaches 0 as n approaches infinity, and we know $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Again, the limit of the ratios is a finite, positive number. The divergence of $\sum_{n=2}^{\infty} \frac{n-1}{n^2+5}$ implies the divergence of the original series of the problem.

(3) Now we use the Ratio Test with the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{\sqrt{n+3}}{2^n}$. Since $a_{n+1} = \frac{\sqrt{n+4}}{2^{n+1}}$, we get

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n+4}}{2^{n+1}}\cdot\frac{2^n}{\sqrt{n+3}} = \lim_{n\to\infty} \frac{1}{2}\sqrt{\frac{n+4}{n+3}} = \frac{1}{2}.$$

The series converges because $1/2 < 1$.

(4) Here we use the Root Test with the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \left(\frac{3n+10}{4n-7}\right)^n$. The computation

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{3n+10}{4n-7} = \frac{3}{4} < 1$$

implies that the series converges.

(5) If $a_n = \dfrac{n!(2n)!5^n}{(3n)!}$ then $a_{n+1} = \dfrac{(n+1)!(2n+2)!5^{n+1}}{(3n+3)!}$ and the ratio is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(2n+2)!5^{n+1}}{(3n+3)!} \cdot \frac{(3n)!}{n!(2n)!5^n} = \frac{5(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}.$$

Now we see

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5}{3} \cdot \frac{2n+2}{3n+2} \cdot \frac{2n+1}{3n+1} = \frac{5}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{20}{27} < 1.$$

The Ratio Test tells us that the series converges.

(6) The inequalities $0 \le |\cos(n^5)| \le 1$ imply $0 \le \dfrac{|\cos(n^5)|}{2^n} \le \dfrac{1}{2^n}$. Since $\sum \frac{1}{2^n} = \sum (1/2)^n$ converges, the Comparison Test tells us that $\displaystyle\sum \frac{|\cos(n^5)|}{2^n} = \sum \left|\frac{\cos(n^5)}{2^n}\right|$ converges. This says that $\displaystyle\sum \frac{\cos(n^5)}{2^n}$ converges absolutely. Therefore, $\displaystyle\sum \frac{\cos(n^5)}{2^n}$ converges.

(7) The function $\frac{1}{x\sqrt{\ln x}}$ is positive, decreasing and continuous for $x \ge 2$. We use the Integral Test. The substitution $u = \ln x$ gives

$$\int \frac{dx}{x\sqrt{\ln x}} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2(\ln x)^{1/2} + C.$$

Therefore,

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{R\to\infty} \int_2^R \frac{dx}{x\sqrt{\ln x}} = \lim_{R\to\infty} 2(\ln R)^{1/2} - 2(\ln 2)^{1/2} = \infty.$$

Since the integral diverges, we conclude that the given series diverges also.

(8) The partial sum $\displaystyle\sum_{n=4}^{N} \frac{1}{(n+3)(n+4)} = \sum_{n=4}^{N} \left(\frac{1}{n+3} - \frac{1}{n+4}\right)$ equals

$$\left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{N+2} - \frac{1}{N+3}\right) + \left(\frac{1}{N+3} - \frac{1}{N+4}\right),$$

which equals $\dfrac{1}{7} - \dfrac{1}{N+4}$. The required sum is the limit as $N \to \infty$ of the partial sums, and that limit is $\frac{1}{7}$.

(9) If we apply the Ratio Test to the series $\sum \dfrac{5^n}{n!}$ then we find

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1.$$

Therefore, $\sum \dfrac{5^n}{n!}$ converges by the Ratio Test. The convergence of this series implies $\lim\limits_{n\to\infty} \dfrac{5^n}{n!} = 0$.

(10) Here we can use rationalization:

$$\begin{aligned}
\lim_{n\to\infty} n\left(\sqrt{n^2+7} - \sqrt{n^2+3}\right) &= \lim_{n\to\infty} n\left(\sqrt{n^2+7} - \sqrt{n^2+3}\right) \cdot \frac{\sqrt{n^2+7} + \sqrt{n^2+3}}{\sqrt{n^2+7} + \sqrt{n^2+3}} \\
&= \lim_{n\to\infty} n \cdot \frac{(n^2+7) - (n^2+3)}{\sqrt{n^2+7} + \sqrt{n^2+3}} \\
&= \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+7} + \sqrt{n^2+3}} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{1+\frac{7}{n^2}} + \sqrt{1+\frac{3}{n^2}}} = \frac{4}{2} = 2.
\end{aligned}$$

(11) We will evaluate $\lim\limits_{x\to\infty} \left(1 - \dfrac{10}{x}\right)^x$. It is easier to evaluate $\lim\limits_{x\to\infty} \ln\left[\left(1 - \dfrac{10}{x}\right)^x\right]$ first. L'Hôpital's Rule implies that $\lim\limits_{x\to\infty} \ln\left[\left(1 - \dfrac{10}{x}\right)^x\right]$ is equal to

$$\lim_{x\to\infty} x \ln\left(1 - \frac{10}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{10}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{10/x^2}{1-\frac{10}{x}}}{-1/x^2} = \lim_{x\to\infty} -\frac{10}{1-\frac{10}{x}} = -10.$$

Applying the exponential function to this, we obtain $\lim\limits_{x\to\infty} \left(1 - \dfrac{10}{x}\right)^x = e^{-10}$. This implies $\lim\limits_{n\to\infty} \left(1 - \dfrac{10}{n}\right)^n = e^{-10}$.

(12) When $f(x) = \sqrt{x} = x^{1/2}$ we get $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = \frac{-1}{4}x^{-3/2}$, $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Therefore, $f(4) = 2$, $f'(4) = 1/4$, $f''(4) = -1/32$, $f^{(3)}(4) = 3/256$. The third Taylor polynomial is

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3.$$

Since $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$, values of u between 4 and 5 will satisfy the condition $|f^{(4)}(u)| \leq \frac{15}{16}4^{-7/2} = \frac{15}{2048}$. We can use $K = \frac{15}{2048}$. The number $\frac{15}{2048} \cdot \frac{|5-4|^4}{4!} = \frac{5}{16384}$ is a bound for $|\sqrt{5} - T_3(5)|$.

(13) Since the Maclaurin series of $\cos x$ begins with

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!},$$

The Maclaurin series of $\cos(x^5)$ begins with

$$1 - \frac{x^{10}}{2!} + \frac{x^{20}}{4!} - \frac{x^{30}}{6!} + \frac{x^{40}}{8!}.$$

Therefore, the Maclaurin series of $x^3\cos(x^5)$ begins with

$$x^3 - \frac{x^{13}}{2!} + \frac{x^{23}}{4!} - \frac{x^{33}}{6!} + \frac{x^{43}}{8!}.$$

(14) Since the binomial series of $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$ begins with

$$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3,$$

the Maclaurin series of $\dfrac{1}{\sqrt{1+x^3}}$ begins with

$$1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9.$$

(15) The Ratio Test with $a_n = \frac{(-3)^n(x-2)^n}{\sqrt{n+4}}$ gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|-3|^{n+1}|x-2|^{n+1}}{\sqrt{n+5}} \cdot \frac{\sqrt{n+4}}{|-3|^n|x-2|^n} = \lim_{n\to\infty} 3|x-2|\frac{\sqrt{n+4}}{\sqrt{n+5}} = 3|x-2|.$$

The power series converges absolutely if $3|x-2| < 1$, which is the same as $5/3 < x < 7/3$. Now we check the endpoints $5/3$ and $7/3$. If $x = 5/3$ then the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n(\frac{5}{3}-2)^n}{\sqrt{n+4}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}},$$

which diverges by the Integral Test. If $x = 7/3$ then the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n(\frac{7}{3}-2)^n}{\sqrt{n+4}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}},$$

which converges by the Leibniz Test. Therefore, the interval of convergence is $5/3 < x \leq 7/3$.

(16) Here we have $f(x) = \dfrac{x^3}{4} + \dfrac{1}{3x}$, $f'(x) = \dfrac{3x^2}{4} - \dfrac{1}{3x^2}$. Note that $1 + (f'(x))^2$ is equal to

$$1 + \left(\left(\frac{3x^2}{4}\right)^2 - \frac{1}{2} + \left(\frac{1}{3x^2}\right)^2\right) = \left(\left(\frac{3x^2}{4}\right)^2 + \frac{1}{2} + \left(\frac{1}{3x^2}\right)^2\right) = \left(\frac{3x^2}{4} + \frac{1}{3x^2}\right)^2.$$

Therefore, the length is

$$\int_1^2 \sqrt{1 + (f'(x))^2}\, dx = \int_1^2 \frac{3x^2}{4} + \frac{1}{3x^2}\, dx = \frac{23}{12}.$$

(17) Using $x'(t) = 3\sin^2 t \cos t$, $y'(t) = -3\cos^2 t \sin t$ we get

$$\begin{aligned} (x'(t))^2 + (y'(t))^2 &= 9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t \\ &= 9\sin^2 t \cos^2 t(\sin^2 t + \cos^2 t) = 9\sin^2 t \cos^2 t. \end{aligned}$$

Since $0 \leq t \leq \pi/4$, we obtain that the length is

$$\int_0^{\pi/4} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{\pi/4} 3\sin t \cos t\, dt = \left.\frac{3\sin^2 t}{2}\right|_0^{\pi/4} = \frac{3}{4}.$$

(18) From $x'(t) = \sin t$ and $y'(t) = 1 - \cos t$ we obtain

$$(x'(t))^2 + (y'(t))^2 = \sin^2 t + 1 - 2\cos t + \cos^2 t = 2(1 - \cos t) = 4\sin^2(t/2).$$

Since $0 \leq t \leq \pi/2$, we conclude $\sqrt{(x'(t))^2 + (y'(t))^2} = 2\sin(t/2)$. The substitution $u = t/2$ allows us to write the required surface area in the form

$$2\pi \int_0^{\pi/2} (t - \sin t) 2\sin(t/2)\, dt = 8\pi \int_0^{\pi/4} (2u - \sin(2u)) \sin u\, du,$$

and this is

$$16\pi \int_0^{\pi/4} u \sin u\, du - 16\pi \int_0^{\pi/4} \sin u \cos u \sin u\, du.$$

The computations

$$\int u \sin u\, du = -u\cos u + \int \cos u\, du = -u\cos u + \sin u + C$$

and

$$\int \sin^2 u \cos u \, du = \frac{\sin^3 u}{3} + C$$

imply that the surface area is

$$16\pi\left(\frac{\sqrt{2}}{2} - \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}\right) - 16\pi\left(\frac{\sqrt{2}}{12}\right) = \left(\frac{20\sqrt{2}}{3} - 2\pi\sqrt{2}\right)\pi.$$

(19) We rotate $y = \sqrt{R^2 - x^2}$, $-R \le x \le R$ about the x-axis. Since $\frac{dy}{dx} = -\frac{x}{\sqrt{R^2-x^2}}$, the required area is

$$\begin{aligned}\int_{-R}^{R} 2\pi\sqrt{R^2-x^2}\sqrt{1+\left(-\frac{x}{\sqrt{R^2-x^2}}\right)^2}\,dx &= \int_{-R}^{R} 2\pi\sqrt{R^2-x^2}\sqrt{\frac{R^2}{R^2-x^2}}\,dx \\ &= \int_{-R}^{R} 2\pi R = 4\pi R^2.\end{aligned}$$

(20) The curve intersects itself when $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$ with $t_1 \neq t_2$. Renaming the t's if necessary, we may assume $t_1 < t_2$. We have to solve for t_1, t_2 in the equations

$$t_1^3 - t_1 = t_2^3 - t_2\ , \quad 4t_1^2 = 4t_2^2.$$

The second equation implies $t_2 = \pm t_1$. Since $t_1 < t_2$, we must have $t_2 = -t_1$ and $t_1 < 0 < t_2$. In view of this, the equation $t_1^3 - t_1 = t_2^3 - t_2$ implies $t_1^3 - t_1 = -t_1^3 + t_1$, which is equivalent to $t_1^3 - t_1 = 0$. Since $t_1 < 0$, we get $t_1 = -1$. In addition, $t_2 = -t_1 = 1$. The curve crosses itself at

$$(x_0, y_0) = (x(-1), y(-1)) = (x(1), y(1)) = (0, 4).$$

Since the slope of every tangent line is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{8t}{3t^2-1},$$

we conclude that the slopes of the two tangent lines at $(x_0, y_0) = (0, 4)$ are -4 and 4, corresponding to $t = -1$ and $t = 1$. Therefore, the two tangent lines are

$$y - 4 = -4x\ , \quad y - 4 = 4x.$$