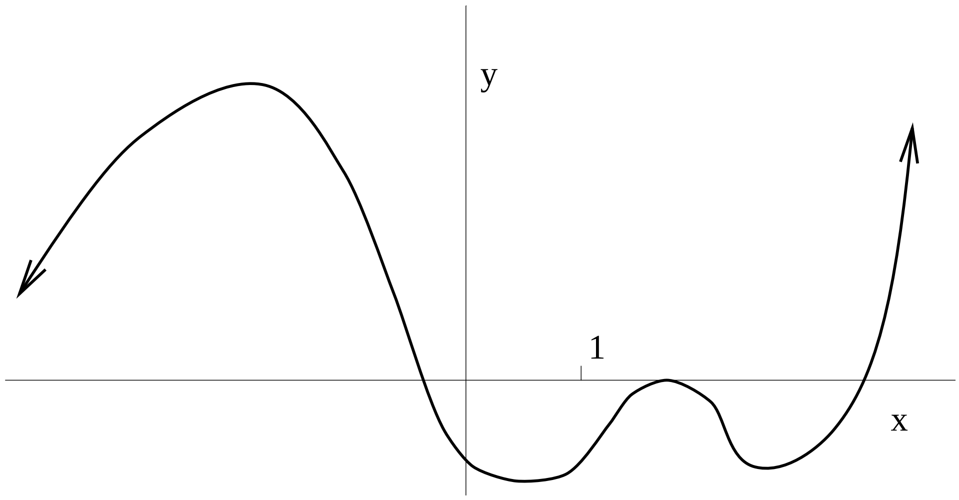
Problems, Week 10

1. The graph of $f'(x)$ is sketched below. Use it to sketch the graph of $f(x)$ as well as you can, assuming in addition that $f(0) = 0$.

You should label this graph, and then use features of it to identify when the graph of f is concave up and concave down, and where it is increasing and decreasing. Locate any local extrema on the graph of f, identifying their nature (max or min). Also, locate any inflection points.



Graph of f'

2. a) Suppose $P(x)$ is any polynomial of degree four which has four distinct real roots. Can $P'(x)$ have fewer than three real roots? Why or why not? How many real roots does $P''(x)$ have?

b) Give an example of a polynomial of degree four which has only three distinct real roots. How many real roots does its derivative have?

3. Suppose you **know** that $g'(x) = \dfrac{2}{1+x^4} - \dfrac{3}{4+x^4}$. Is $g(0) < g(1)$?

Please note: It is not likely at this time that you can write a formula for a g with this derivative (that can be done, and such g's have very complicated formulas). So you will have to make some *indirect* argument, just using the information you have about g'.

Note: here is such a function:

$$g(x) = \frac{\sqrt{2}}{4} \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x-1)$$

$$+\frac{3}{16}\ln(x^2-2x+2) - \frac{3}{8}\arctan(x-1) - \frac{3}{16}\ln(x^2+2x+2) - \frac{3}{8}\arctan(x+1)$$

Does knowing this formula help or is studying the derivative easier?

4. a) Suppose you know that $h'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$. What are the critical numbers of h? Which of them are local extrema, and what kind of local extrema are they?

b) Suppose you know that $k'(x) = x(x-1)^{2/3}(x-2)^{3/5}(x-3)^{4/7}$. What are the critical numbers of k? Which of them are local extrema, and what kind of local extrema are they?

Note: you are *not* asked to compute h and k explicitly.

5. Suppose

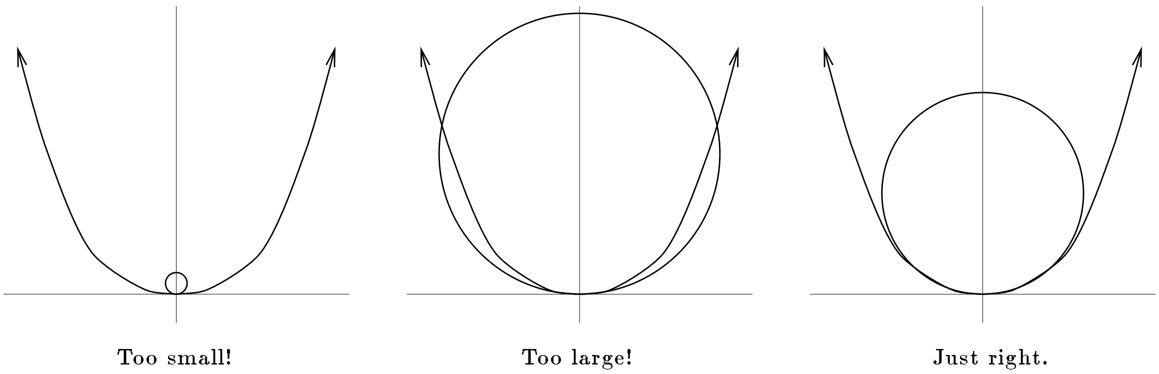
$$5x^3y - 3xy^2 + y^3 = 6 \ .$$

Note that $(1,2)$ is a point on this curve. Is the curve concave up or concave down at $(1,2)$?

Explicit way to go y *can* be solved as a function of x.* Then you can differentiate the formula twice and evaluate when $x = 1$.

Implicit way to go Find $\dfrac{dy}{dx}$ implicitly and then differentiate again to get $\dfrac{d^2y}{dx^2}$. Evaluate everything at $(1,2)$.

6. Find the magical circle which is largest and which just fits "inside" the parabola $y = x^2$.



The parabola $y = x^2$ and some inside circles tangent at $(0,0)$

7. a) Suppose f is differentiable on the interval $a \le x \le b$ and that there are numbers α and β so that if x is in that interval, $\alpha \le f'(x) \le \beta$. Find reasonable upper and lower bounds for $f(b)$ depending on $f(a)$, α, and β. That is, find L and U so that $L \le f(b) \le U$, and describe L and U in terms of $f(a)$, a, b, α, and β.

b) The fuel consumption of an automobile varies between 17 and 23 miles per gallon, according to the driving conditions. Let $f(x)$ be the number of gallons of fuel left in the

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* Here it is (really!):

$$y = \left(-\frac{5}{2}x^4 + 3 + x^3 + \frac{1}{18}\sqrt{1500x^9 - 675x^8 - 4860x^4 + 2916 + 1944x^3}\right)^{1/3}$$
$$- \frac{\frac{5}{3}x^3 - x^2}{\left(-\frac{5}{2}x^4 + 3 + x^3 + \frac{1}{18}\sqrt{1500x^9 - 675x^8 - 4860x^4 + 2916 + 1944x^3}\right)^{1/3}} + x$$

tank after x miles have been driven. If $f(100) = 15$, find upper and lower estimates for $f(200)$. (Note: you may assume you are driving a differentiable automobile.)

8. a) Suppose f is differentiable on the interval $a \le x \le b$ and that there are numbers α and β so that if x is in that interval, $\alpha \le f'(x) \le \beta$. Find reasonable upper and lower bounds for $f(b)$ depending on $f(a)$, α, and β. That is, find L and U so that $L \le f(b) \le U$, and describe L and U in terms of $f(a)$, a, b, α, and β.

b) Suppose you know that $f'(x) = \frac{1}{\sqrt{1+x^2}}$, and that $f(0) = 3$. **Estimate** $f(1)$ (that is, find over- and underestimates for $f(1)$ based on the information given).

c) If $f(x)$ is the same function as in part b), find an estimate for $f\left(\frac{1}{2}\right)$. **Use** your estimate for $f\left(\frac{1}{2}\right)$ to obtain an estimate for $f(1)$. (Note: you must adjust the a and b and α and β you use here carefully; you will be doing the process twice.)