# Problems, Week 13

1. I looked up the sum of the first $n$ $9^{\text{th}}$ powers. That is, if you define

$$S_9(n) = 1^9 + 2^9 + 3^9 + \cdots + n^9$$

(in summation notation, $S_9(n) = \sum_{k=1}^{n} k^9$) then an explicit formula for $S_9(n)$ is:

$$S_9(n) = \frac{1}{10}(n+1)^{10} - \frac{1}{2}(n+1)^9 + \frac{3}{4}(n+1)^8 - \frac{7}{10}(n+1)^6 + \frac{1}{2}(n+1)^4 - \frac{3}{20}(n+1)^2$$

(It isn't even clear that this formula gives an integer!) **Assume that this formula is true.**

a) Check the formula for $S_9$ by evaluating it for $n = 3$, and seeing that the answer is the same as $1^9 + 2^9 + 3^9$. (*Please* use a calculator – I did!)

b) Find some definite integral and some approximating sum which knowledge of this formula will allow you to evaluate exactly when $n \to \infty$. Write out the approximating sums, and evaluate the limit of these sums.

2. Let $f(x)$ be defined on $0 \le x \le 1$ by the following rule:

$f(x)$ is the first digit in the decimal expansion for $x$.

For example, $f(1/2) = 5$ and $f(0.719) = 7$.

a) Sketch the graph of $y = f(x)$. Be careful to use an appropriate scale for $x$ and for $y$.

b) Use the geometric definition of the definite integral to find $\int_0^1 f(x)\,dx$.

c) Now let the function $g(x)$ be defined by

$g(x)$ is the second digit in the decimal expansion for $x$.

(For example, $g(0.437) = 3$.) Compute $\int_0^1 g(x)\,dx$.

3. Water is flowing through mighty Mill Brook (a local stream located only a short distance from where this was written). Two engineering students (Alpha and Beta) are assigned the task of estimating the water flow through the brook between 8 AM and noon. They are provided with instruments which will allow them to tell what the water flow is at any time in **gallons per minute**.

a) Lazy student Alpha makes observations of the water flow every hour on the hour from 8 AM to noon. Here are Alpha's observations:

| Time | Flow, in Gals/Min | |
|---|---|---|
| 8:00 | 5.6 | |
| 9:00 | 8.9 | (Thunderstorm!) |
| 10:00 | 12.4 | |
| 11:00 | 7.2 | (Lunch break.) |
| 12:00 | ...... | |

What is Alpha's approximation for the total water drained through Mill Brook during those 4 hours?

b) Student Beta, more energetic, checks the water flow every *half* hour. Here's Beta's data:

| Time | Flow, in Gals/Min | |
|---|---|---|
| 8:00 | 5.6 | |
| 8:30 | 7.3 | |
| 9:00 | 8.9 | |
| 9:30 | 8.3 | (Thunderstorm!) |
| 10:00 | 12.4 | |
| 10:00 | 11.2 | |
| 11:00 | 7.2 | |
| 11:00 | 6.1 | (Lunch break.) |
| 12:00 | ...... | |

What is Beta's estimate of the total water flow through Mill Brook during the period 8 AM to noon?

4. An alien spaceship is found. A major component of the spaceship is a thin metal bar, 120 meters long*. It is very **very** heavy. A metallurgist† takes small samplings of the bar at 20 meter intervals, and finds the density (in kilograms per centimeter). Here is her

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* Metric aliens

† Spelled right, because I looked it up.

data:

| Meter mark | Density | |
|---|---|---|
| 0 | 30.4 | |
| 20 | ... | Data lost. (Stolen by aliens?) |
| 40 | 46.5 | |
| 60 | 65.8 | |
| 80 | 29.2 | |
| 100 | 52.1 | |
| 120 | ... | Lunch break. |

Estimate the total weight of the alien object.