Problems, Week 2

Please write solutions to one of these problems and hand them in next Monday. These written solutions should be of high-quality, with complete explanations given. While I encourage you to discuss the problems with other students and with your instructors, the written work you hand in must be your own explanations of the problems you choose.

1. a) The function S (the "squaring function") has domain all of $\mathbb{R}$ (all real numbers) and its values are given by $S(x) = x^2$ for all x. Now consider the function T whose domain is also all of $\mathbb{R}$ and which is defined by

$$T(x) = \begin{cases} S(x) & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$

Sketch a graph of T. What is $\lim_{x \to 5} T(x)$ and what is $\lim_{x \to 3} T(x)$? Give some reason to believe your assertions.

b) Suppose S is again the squaring function defined above. Now an evil interstellar visitor comes and *changes* exactly *one million values* of S and thus creates a new function, V. What can you say about $\lim_{x \to a} V(x)$ for all values of a? Give reasoning to support your assertions.

2. The profile $P(x)$ of a mountain road's height in miles is given by the following (x is measured in flat miles from the start of the road):

$$P(x) = \begin{cases} 3 + .04\cos(2x) & \text{if } 0 \leq x \leq 10 \\ \textbf{Giant Valley!} & \text{if } 10 < x < 10.04 \\ 3.02 + .005(x - 11) & \text{if } 10.04 \leq x \leq 12 \end{cases}$$

a) Sketch the profile of the road.

b) Give the height of a straight bridge across the valley so that vehicles will not fall down. Make and verify an assertion about limits of the resulting profile function at $x = 10$ and $x = 10.04$.

c) (More subtle!) Do you think that drivers of the vehicles will feel undesirable shocks as they cross the bridge? What additional constraints on the bridge design would you suggest so that the shocks would not be as severe?

3. Use your calculator to guess the limiting behavior as $x \to 0$ of these functions with domain indicated:

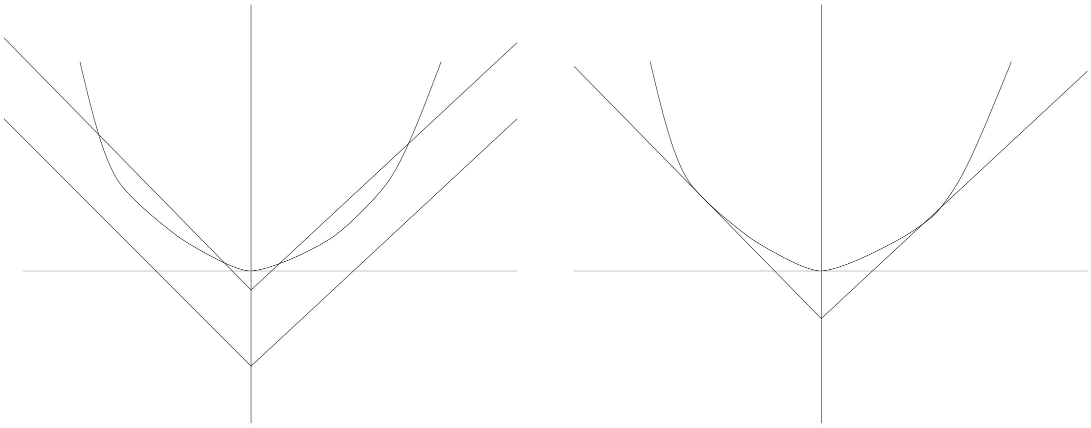
a) $F_1(x) = (1 + x)^{1/x}$, with domain $(-1, 0)$ and $(0, +\infty)$.

b) $F_2(x) = \left(1 + x^2\right)^{1/x}$, with domain $(-\infty, 0)$ and $(0, +\infty)$.

c) $F_3(x) = \left(1 + \sqrt{|x|}\right)^{1/x}$, with domain $(-\infty, 0)$ and $(0, +\infty)$.

(You may want to consider behavior as $x \to 0$ from each side, especially for F_3.) Could you anticipate the answers?

4. Find the magical vertical translation of $y = |x|$ that just "supports" the parabola $y = x^2$.



The parabola $y = x^2$ and some vertical translates of $y = |x|$

Hint: you want to find c so that $y = |x| + c$ just touches $y = x^2$, and you may as well consider only the right-hand side of the picture first.

Low-tech way to go Find c so that there's exactly *one* common point shared by $y = x+c$ and $y = x^2$.

High-tech way to go Find c so that when $y = x+c$ and $y = x^2$ touch, the line is *tangent* to the parabola.

5. After winning her championship softball game with a no-hitter, the winning pitcher throws the game ball directly up into the air. It moves according to the law $s(t) = 5 + 96t - 16t^2$, where t is the time in seconds after it is thrown, and $s(t)$ is the height in feet above the ground at time t. Find:

a) the velocity of the ball after 1.5 seconds.

b) the maximum height the ball reaches.

c) the average speed of the ball while it is rising.

6. For each of the three cases below, sketch a graph of a function that satisfies the stated conditions.

a) $\lim_{x \to 2} f(x) = 3$ and $f(2) = 4$.

b) $\lim_{x \to 0} f(x)$ does not exist, and $|f(x)| < 2$ for all x .

c) $\lim_{x \to 0} f(x) = f(0) + 1$.