Problems, Week 3

1. A rational function is a quotient of two polynomials. Simple examples of rational functions are $x+1$ and $\dfrac{(3x+7)(6x-1)}{(x^2+6)(x-11)}$ and $\dfrac{2.03x^{78}+3.7x^{45}-.09}{4.5x^{99}-\sqrt{13}x}$.

a) (Easy) Find a rational function R whose "natural" domain is all real numbers *except* 1 and 2, and so that for all x in its domain, $R(x)=x^2$. Sketch a graph of R.

b) (Harder) Find a rational function T with all of the following properties. Sketch a graph of T.

i) The natural domain of T is all real numbers *except* for 1 and 3 and 4.

ii) $\lim\limits_{x\to 1^-} T(x)=+\infty$ and $\lim\limits_{x\to 1+} T(x)=+\infty$;
$\lim\limits_{x\to 3^-} T(x)=-\infty$ and $\lim\limits_{x\to 3+} T(x)=+\infty$;
$\lim\limits_{x\to 4^-} T(x)=+\infty$ and $\lim\limits_{x\to 4+} T(x)=-\infty$;
$\lim\limits_{x\to +\infty} T(x)=+\infty$ and $\lim\limits_{x\to -\infty} T(x)=+\infty$.

iii) T has roots only at 0 and 2 and 5.

2. a) Suppose $F(x)=x^3$ and $G(x)=4\cos(7x+5)+8\sin(x^2-9)+6$. What are $\lim\limits_{x\to+\infty} F(x)$ and $\lim\limits_{x\to-\infty} F(x)$? Find numbers A and B so that all values of G are between A and B (that is, $A<G(x)<B$ for all x). The values of A and B don't have to be precise!

b) Make a rough sketch on the same graph of $y=F(x)$ and $y=G(x)$ and $y=A$ and $y=B$.

c) Find one root of $F(x)=G(x)$ approximately.

d) Prove the following result: **if** F and G are continuous functions defined on $\mathbb{R}$ (all real numbers) and **if** $\lim\limits_{x\to+\infty} F(x)=+\infty$ and $\lim\limits_{x\to-\infty} F(x)=-\infty$ and **if** G is bounded (so there are numbers A and B so that $A<G(x)<B$ for all x) then the equation $F(x)=G(x)$ **must** have at least one root. (Hint: this isn't difficult. Look up the Intermediate Value Theorem.)

3. One of the classical sayings from the days of sailing ships is

Constant bearing means collision.

a) The statement is true when velocity is constant, and it can be illustrated with the following example:

Ship A sails on the $x-$axis, and its position at time t is $\big(5(4-t)+k,0\big)$. Ship B sails on the line $y=x$, and its position at time t is $\big(3(4-t),3(4-t)\big)$. The *bearing* at time t from B to A is defined to be the slope of the line segment from B to A at time t (this makes sense only when they *don't* collide!). Show that the ships collide *exactly* when the bearing is constant as a function of time. (Which value of the parameter k will make them collide and at what time will the collision occur? Which value of k gives the ships constant bearing?)

b) The statement is more complicated and not true when velocity isn't constant:

Again, suppose ship A sails on the x−axis, and its position at time t is $\left(5(4-t)+k, 0\right)$. Now suppose ship B, sailing again on the line $y = x$, has its position at time t given by $\left((t-4)^2, (t-4)^2\right)$*. Calculate the bearing from B to A and show that, by selecting k properly, the ships can collide even when the bearing *isn't* constant. In the case of a collision, what is the limiting value of the bearing from B to A? Sketch a graph of the bearing from B to A from time $t = 0$ until the collision.

4. a) Find a number $d_1 > 0$ such that whenever $|x-1| < d_1$ it follows that $|x+1| < 4$.

b) Find a number $d_2 > 0$ such that whenever $|x-1| < d_2$ it follows that $4|x-1| < 1/10$.

c) Find a number $d_3 > 0$ such that whenever $|x-1| < d_3$ it follows that $4|x-1| < 1/100$.

d) Let $\varepsilon > 0$ be given. In terms of ε, find a number $d > 0$ such that whenever $|x-1| < d$ it follows that $4|x-1| < \varepsilon$.

e) Let $\varepsilon > 0$ be given. In terms of ε, find a number $\delta > 0$ such that whenever $|x-1| < \delta$ it follows that $|x+1||x-1| < \varepsilon$.

f) Prove from the definition of limit that $\lim_{x \to 1} x^2 = 1$.

5. Questions about asymptotic growth "near" ∞ occur naturally when computer scientists analyze algorithms. One seemingly simple problem is sorting. How many comparisons are required to sort a list of n numbers? One reference † gives the following average running times for several sorting algorithms as a function of n:

Name	**Running time**
Comparison	$4n^2 + 10n$
Merge exchange	$3.7n(\ln n)^2$
Heapsort	$23.08n \ln n + 0.2n$

Which sorting method would you rather use if, in your application, $10 \le n \le 20$ (e.g., sorting a bridge hand)? Which would you rather use if, in your application, $100 \le n \le 150$ (e.g., sorting grades in a lecture course)? Which would you rather use if $n \approx 10^6$ (e.g., sorting license plate numbers in New Jersey)? What happens to these functions as $n \to \infty$? Later we will be able to be more precise about methods to compare rates of growth.

6. Let

$$f(x) = \begin{cases} 3x-2, & \text{if } x < 0 \\ ax+b, & \text{if } 0 \le x \le 1 \\ 3x+4, & \text{if } x > 1 \end{cases}.$$

Find a and b so that $f(x)$ is continuous for all values of x.

* We'll see later that this sort of motion has non-zero acceleration.

† *Sorting and Searching*, volume 3 of *The Art of Computer Programming*, by Donald Knuth.