Problems, Week 4

1. In this problem, $C(x)$ is the function given by $C(x) = 1 - x^2$ with domain $0 \le x \le 1$.

a) Plot $y = C(x)$ and $y = x^{17}$ when $0 \le x \le 1$ on the same graph. Find the tangent lines to these graphs at their unique intersection point. Plot the tangent lines on this graph also.

b) Plot $y = C(x)$ and $y = x^{1/17}$ when $0 \le x \le 1$ on the same graph. Find the tangent lines to these graphs at their unique intersection point. Plot the tangent lines on this graph also.

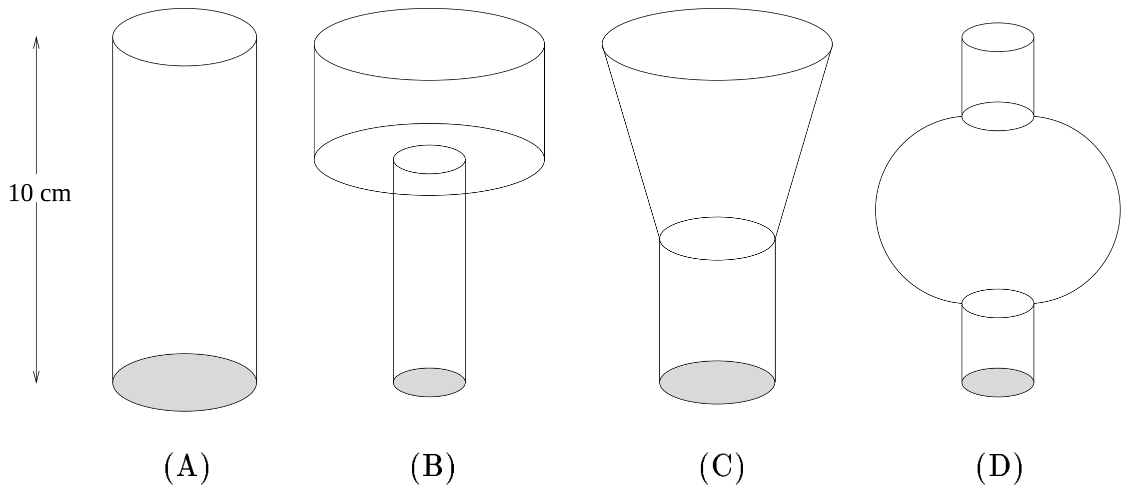
c) Find a differentiable function whose domain is $0 \le x \le 1$, which satisfies $f(0) = 0$ and $f(1) = 1$, whose graph intersects the graph of $y = C(x)$ at exactly *one* point, and so that, at this intersection point, the tangent lines to the two graphs are perpendicular.

> "Captain, we must travel from our current position to meet the Ambassador's ship at position VBL-1290. To do this, we must leave the galaxy, which is perilous and energy-draining. I recommend that we exit orthogonal to the galactic interface since that will minimize energy loss." "Make it so."

2. a) Let $g(x) = |x^3 - 1|$. Evaluate $g'(-1)$, $g'(0)$, and $g'(1)$.

b) Let $f(x) = (3x - 1)^{1/2}$. Express $f'(1)$ as a limit and evaluate the limit.

3. Four containers are each 10 cm tall, and each of them has a volume of 30 cm^3. They are each being filled by a liquid at the rate of 5 cm^3 per minute. Here is a picture of the four containers:



a) For each of the containers, graph the height, $h(t)$, of the level of the liquid in the containers measured in centimeters as a function of time, t, measured in minutes.

b) Which of the functions graphed in a) are continuous? Justify your conclusions as well as you can.

c) Which of the functions graphed in a) are differentiable? Justify your conclusions as well as you can.

Special note: this liquid is *ideal*, so please assume there's no surface tension and no bubbles and no magic and ...

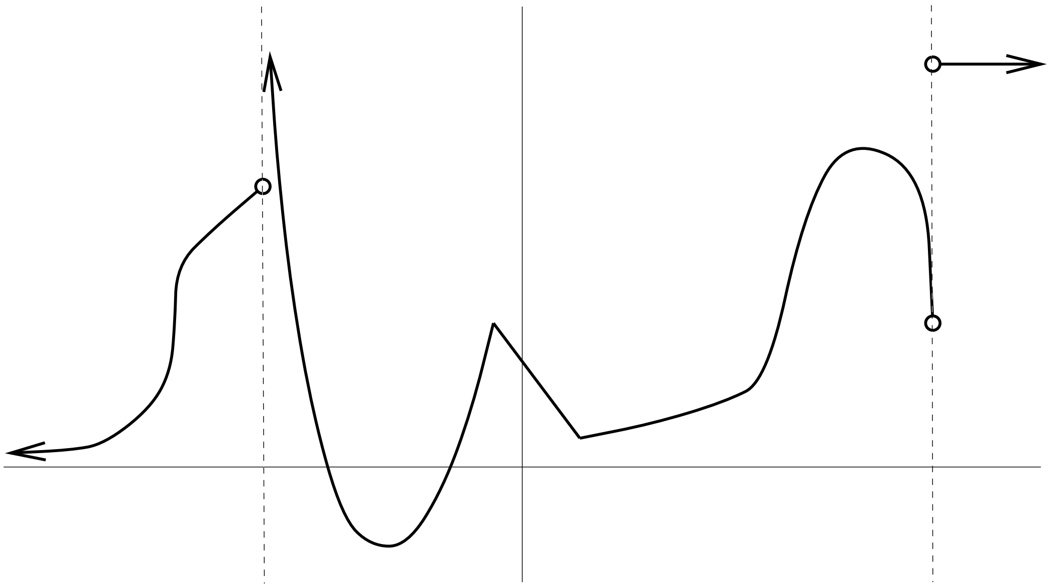
4. a) Show from the definition of derivative that

i) $f(x) = x|x|$ is differentiable at $x = 0$.

ii) $g(x) = (1 + |x|)^{1/2}$ is *not* differentiable at $x = 0$.

b) Graph each function carefully near $x = 0$ and discuss how the graphs appear to confirm the results of a).

5. Below is the graph of a function, F, which has *interesting* behavior. Label any points on the graph which occur in your discussion of the questions below.



Graph of $y = F(x)$

a) Where is F not continuous? Why?

b) Where is F not differentiable? Why?

c) Sketch a graph of F' wherever F' is defined.

6. Find all values of p for which the following functions are

a) continuous everywhere or

b) differentiable everywhere.

i) $f(x) = \begin{cases} x^2 + p, & \text{if } x \geq 0 \\ 4 - x^2, & \text{if } x < 0 \end{cases}$ ii) $g(x) = \begin{cases} px^2 + x, & \text{if } x \geq 0 \\ x^3 + p^2 x, & \text{if } x < 0 \end{cases}$

c) Graph every function resulting from the special values of p found in answer to a) or b).