Problems, Week 9

1. Graph the functions $F(x) = \dfrac{4x}{x^2+1}$ and $G(x) = 2\sin(2\arctan x)$ in the same viewing window over the interval $[-5, 5]$. Explain what you see.

2. A surveyor stands on flat ground at an unknown distance from a tall building. She measures the angle from the horizontal ground to the top of the building; this angle is $\pi/3$. Next she paces 40 feet further away from the building. The angle from the ground to the top of the building is now measured to be $\pi/4$.

a) How tall is the building?

b) If the surveyor moves 20 feet further away from the building, what will be the angle from the horizontal ground to the top of the building?

3. Particle **A** moves left along the line $y = 2$ at 3 units/sec. Particle **B** moves down along the line $x = 4$ at 5 units/sec. They both start at time 0 at the point $(4, 2)$. An observer, **O**, is located at $(0, 0)$.

$\rightarrow$ Choose one of the following. This problem turns out to be amusingly (!) intricate. $\leftarrow$

At what rate (and in what units) is the angle that **O** sees between **A** and **B** changing at the instant that the angle is 10°? (You'll need to convert 10° to radians if you analyze this situation using any of the conventional functions.)

a) How fast (in radians/sec) is the angle that **O** sees between **A** and **B** changing at the instant that the angle is a right angle?

b) How fast (in radians/sec) is the angle that **O** sees between **A** and **B** changing at the instant that the angle is a straight angle?

ã) How fast (in radians/sec) is the angle that **O** sees between **A** and **B** changing at the instant that the distance between **A** and **B** is 10 units?

$\tilde{\text{b}}$) How fast (in radians/sec) is the angle that **O** sees between **A** and **B** changing at the instant that the distance between **A** and **B** is 100 units?

4. a) Find the following limits:

$$\lim_{x\to 0} \frac{\sin x}{x} \qquad \lim_{x\to 0} \frac{(\sin x) - x}{x^3} \qquad \lim_{x\to 0} \frac{(\sin x) - \left(x - \frac{x^3}{6}\right)}{x^5}$$

b) Do you detect a pattern? Find what should be put in place of "?" in order to have a finite non-zero value for

$$\lim_{x\to 0} \frac{(\sin x) - \left(x - \frac{x^3}{6} + (?)\right)}{x^7}$$

5. Find the following limits:

a) $\lim_{x \to 0} \frac{\arcsin x - x}{x^3}$

b) $\lim_{x \to 0} \frac{\arctan x - x}{x^3}$

6. Find the limits for the following indeterminate forms of the type "$\infty - \infty$".

a) $\lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x}$

b) $\lim_{x \to 0} \frac{1}{x^2} - \frac{1}{x}$

c) $\lim_{x \to 0} \frac{1+x}{x} - \frac{1-x}{x}$

Note that three completely different types of behavior occur.

7. For each of the following pairs of functions, determine which grows faster as $x \to \infty$, or if they grow at the same rate.

a) e^x and 2^x

b) e^x and x^2

c) e^x and e^{2x}

d) x^2 and $x^{1/2}$

e) $e^{1/x}$ and $e^{1/(2x)}$

f) $\exp(x^2)$ and $\exp(x^3)$

g) x^2 and $\ln(2x)$

h) x^2 and $x^2 + 2x$

8. For each of the following pairs of functions, determine which grows faster as $x \to 0^+$, or if they grow at the same rate.

a) $e^{1/x}$ and $2^{1/x}$

b) $e^{1/x}$ and $x{-}2$

c) $e^{1/x}$ and $e^{1/(2x)}$

d) x^{-2} and $x^{-1/2}$

e) $e^{1/x}$ and $-\ln(2x)$

f) $\exp(1/x^2)$ and $\exp(1/x^3)$

g) x^{-2} and $-\ln(2x)$

h) x^{-2} and $(x^{-2} + 2x^{-1})$