October 7, 1995

Answers to Exam #1 for Math 151, sections 31 & 32

Here are the questions together with some answers to the first exam. There are certainly other methods of getting the answers to some of the questions. I hope that these will be useful to you. Please note: I have added occasional **Comments** not really needed for the solutions which might be interesting or helpful.

(15) 1. Suppose that $f(x) = x^3 + 2x + 1$ and that g is a differentiable function about which the following is known:

$$g(1) = 3 \qquad\qquad g'(1) = 7$$

Carefully compute the following quantities (an answer alone will *not* receive full credit):

$$(f+g)'(1) \qquad\qquad (f\cdot g)'(1) \qquad\qquad \left(\frac{f}{g}\right)'(1)$$

Answer Of course since $f(x) = x^3 + 2x + 1$, $f(1) = 4$ and $f'(x) = 3x^2 + 2$ so $f'(1) = 5$. Therefore:
$(f+g)'(1) = f'(1) + g'(1) = 5 + 7 = 12$
$(f\cdot g)'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1) = 5\cdot 3 + 4\cdot 7 = 43$
$\left(\frac{f}{g}\right)'(1) = \dfrac{f'(1)\cdot g(1) - g'(1)\cdot f(1)}{(g(1)^2)} = \dfrac{5\cdot 3 - 4\cdot 7}{3^2} = \dfrac{-13}{9}$

(15) 2. a) Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.

Answer I'll use the following form of the definition: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.
If $f(x) = \dfrac{1}{\sqrt{x}}$ then

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\frac{\sqrt{x}-\sqrt{x+h}}{sqrtx\sqrt{x+h}}}{h} = \frac{\sqrt{x}-\sqrt{x+h}}{h\cdot\left(\sqrt{x}\sqrt{x+h}\right)}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} =$$
$$= \frac{x-(x+h)}{h\cdot\left(\sqrt{x}\sqrt{x+h}\right)\cdot\left(\sqrt{x}+\sqrt{x+h}\right)} = \frac{-h}{h\cdot\left(\sqrt{x}\sqrt{x+h}\right)\cdot\left(\sqrt{x}+\sqrt{x+h}\right)} =$$
$$= \frac{-1}{\left(\sqrt{x}\sqrt{x+h}\right)\cdot\left(\sqrt{x}+\sqrt{x+h}\right)}$$

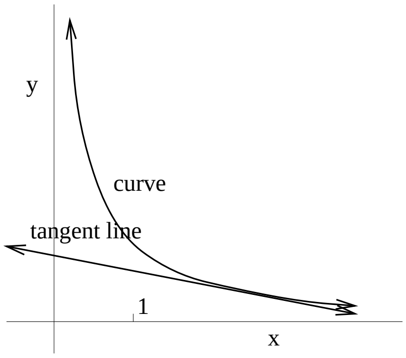
and this last algebraic form is continuous at $h = 0$ so I can evaluate the limit as $h \to 0$ by just "plugging in" which gives $f'(x) = \dfrac{-1}{\left(\sqrt{x}\sqrt{x}\right)\cdot\left(2\sqrt{x}\right)}$.
Comment An interested student could notice, but *not* during the exam, that we've verified the derivative of $x^{-.5}$ is $-.5x^{-1.5}$.

b) Use your answer to a) to find an equation for the line tangent to the curve $y = \dfrac{1}{\sqrt{x}}$ when $x = 4$.

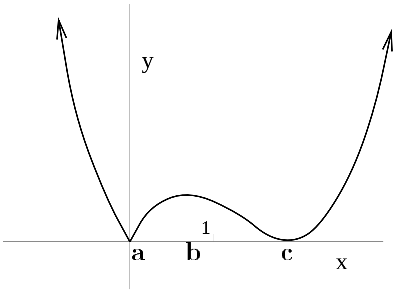
Answer If $x = 4$ then $y = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$. And $f'(2) = \dfrac{-1}{\left(\sqrt{4}\sqrt{4}\right)\cdot\left(2\sqrt{4}\right)} = \dfrac{-1}{16}$ so an equation of the tangent line is $\left(y - \frac{1}{2}\right) = \dfrac{-1}{16}(x-4)$.

c) Sketch the graph of $y = \dfrac{1}{\sqrt{x}}$ *and* the line you found in answer to b).

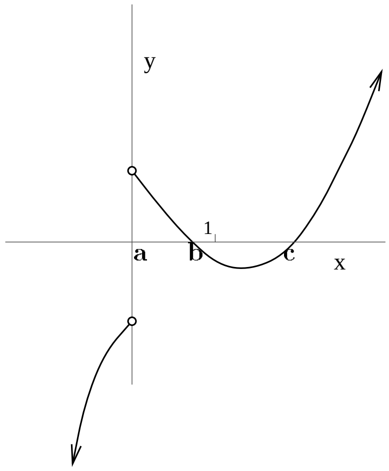
Answer



(15) 3. Here are graphs of the functions F and G:



Graph of $y = F(x)$



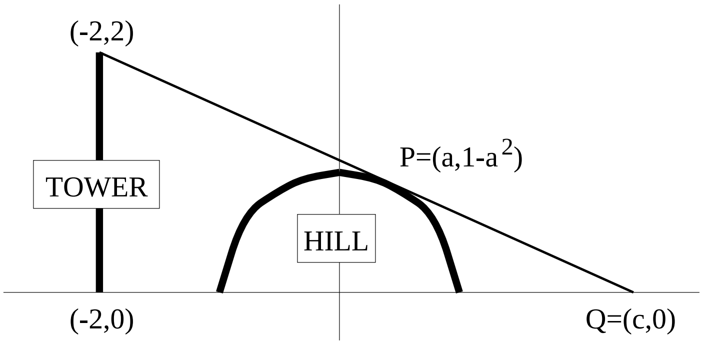
Graph of $y = G(x)$

One of these functions is the derivative of the other function. Identify which is the derivative and explain your answer as well as you can with reference to the graphs (which you may wish to label). Your answer should be written in complete English sentences with as much evidence as possible given to support your assertion.

Answer Since a differentiable function is continuous, and G is not continuous (because of the jump at **a**), F cannot be the derivative of G. G is the derivative of F because G is 0 at several places (**b** and **c**) where the tangent line to F is horizontal. G is the derivative of F because F's graph has a corner at one place (**a**) and G does not exist at that value of x. G is the derivative of F because F's tangent lines are tilted downward in several regions (where $x < 0$ and between **b** and **c**) and in those regions, G's values are negative. G is the derivative of F because F's tangent lines are tilted upward in several regions (between **a** and **b** and for x's larger than **c**) and in those regions, G's values are positive.

Comment What I wrote above is more than necessary for full credit on this problem. But I will want several reasons. I will want them expressed clearly in sentences, referring to the graphs. This referencing will be simpler for me to understand if you label the graphs as I requested. It is certainly not necessary to discuss every feature of the graphs in order to get full credit on this problem. If you are curious about whether a function given by a formula can create a graph like f's above, look at a graph of $|x|(x-2)^2$.

(20) 4. The $x-$axis represents the flat ground. There's a hill whose profile is given by $y = 1 - x^2$ for $-1 \le x \le 1$. There's a tower whose base is at $(-2, 0)$ and whose top is at $(-2, 2)$. How close to the origin on the positive $x-$axis can you come and still see (just barely) the top of the tower?



Above is a picture which you might find helpful. You may be guided in your work by the following:

a) Find the number a with $0 < a < 1$ so that the line connecting $(-2, 2)$ and $P = (a, 1 - a^2)$ is tangent to the parabola.

Answer The derivative of $1 - x^2$ is $-2x$, telling us the slope of the tangent line to the parabola at the point P is $-2a$. But the tangent line also passes through $(-2, 2)$ and $(a, 1 - a^2)$, so that its slope must also be $\dfrac{2-(1-a^2)}{-2-a} = \dfrac{1+a^2}{-2-a}$. Therefore we know $-2a = \dfrac{1+a^2}{-2-a}$, which leads to $2a^2 + 4a = 1 + a^2$. To solve for a we rewrite and get $a^2 + 4a - 1 = 0$. This can be solved (in various ways) to get $a = \dfrac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5} \approx -4.236$ or $+.236$. I'll choose the positive solution because the geometry of the problem forces me to.

b) Find the intersection point, $Q = (c, 0)$, of the $x-$axis and the line you found in a).

Answer The slope of the line connecting $(-2, 2)$ and P is $-2a$ which is $-2(+.236) = -.472$. But the slope of the line connecting $Q = (c, 0)$ with $(-2, 2)$ is $\dfrac{2-0}{-2-c}$. Since these are equal, we know $\dfrac{2-0}{-2-c} = -.472$. We can solve for c: $2 = .944 + .472c$ gives $1.056 = .472c$ so $c \approx 2.237$ and the point we are looking for seems to be $(2.237, 0)$.

Comment I think this is a difficult problem. I graphed $y = 1 - x^2$ and $y = -.472(x + 2) + 2$ together (the second equation's graph is a straight line going through the point $(-2, 2)$ with slope $-.472$) and the result seems to look like the picture above. The line does cross the $x-$axis at the point indicated. Humility and honesty compel me to admit that I didn't do the arithmetic correctly the first *two* times I did the problem, even with electronic help!

(20) 5. Consider the function

$$H(x) = \cos(Ax) + Ax^5$$

Note that the parameter A appears in *two* places in the formula for H.

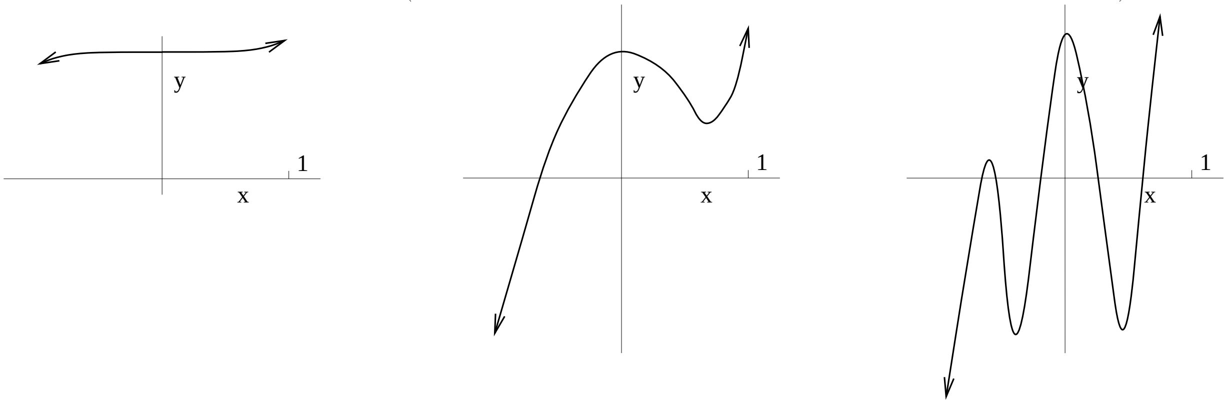
a) If $A = .1$ how many roots do you think $H(x) = 0$ has in the interval $[-1, 1]$? Why?

Answer $H(x) = 0$ seems to have *no* roots in the interval $[-1, 1]$. A graph shows that it seems to be always positive (with values greater than about .89).

Comment More evidence than this will not be necessary, but $\cos(\pm.1)$ is still fairly close to 1, and $.1(\pm1)^5$ won't move it much. The values of H between the endpoints don't wiggle very much. The vague use of the word "wiggle" here shows me that I'm not being very precise. We will get better language and more convincing arguments. Look at the picture on the next page, please.

b) If $A = 100$ explain why general theory guarantees that $H(x) = 0$ *must* have a root in $[-1, 1]$. (Hint: consider H's values at the endpoints of the interval.)

Answer Here $H(x) = \cos(100x) + 100x^5$. $H(-1) = \cos(-100) - 100 \approx -99.13$ and $H(1) = \cos(100) + 100 \approx 100.86$. Since H is a continuous function, and its value at one end of the interval is negative and its value is positive at the other end, the Intermediate Value Theorem assures us that $H(x) = 0$ *must* have at least one root in the interval.
Comment Close inspection of a graph convinces me that the equation probably has 25 roots in the interval.
c) Locate one approximate solution with accuracy $\pm.01$ of the equation $H(x) = 0$ in the interval $[-1, 1]$ when $A = 2$.
Answer Looking closely at a picture or using `solve` under the `math` menu on a TI-82 gives $-.659$ as an approximate root. Acceptable answers would thus seem to be between $-.649$ and $-.669$. Since I get confused about whether to round or truncate, acceptable answers will be any number in the interval $[-.67, -.64]$.
Comment Here are some pictures ($A = 100$ is much too fuzzy to draw clearly on the same scale):

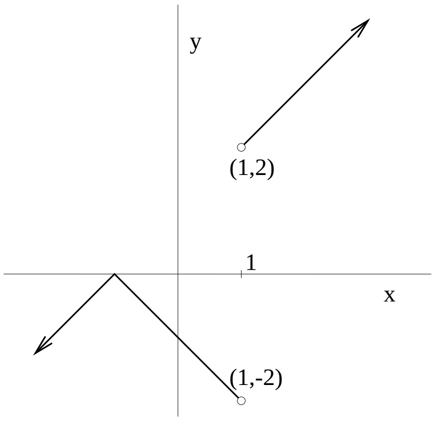


Graph of $y = H(x)$ with various A's in the same "window"
with $A = .1$ with $A = 2$ with $A = 10$ – note more wiggles!

(15) 6. Let $F(x) = \dfrac{|x^2 - 1|}{x - 1}$.

a) Find
i) $\lim_{x \to 1^+} F(x)$ ii) $\lim_{x \to 1^-} F(x)$

Answer i) If $x > 1$ then $x^2 > 1$ and $x^2 - 1 > 0$ so $|x^2 - 1| = x^2 - 1$. Thus as $x \to 1^+$, $F(x) = \dfrac{|x^2 - 1|}{x - 1} = \dfrac{x^2 - 1}{x - 1}$ $= x + 1$ and the limit as $x \to 1$ of the last expression is 2.
Answer ii) If $0 < x < 1$ then $x^2 < 1$ and $x^2 - 1 < 0$ so $|x^2 - 1| = -(x^2 - 1) = 1 - x^2$. Thus as $x \to 1^+$, $F(x) = \dfrac{|x^2 - 1|}{x - 1} = \dfrac{1 - x^2}{x - 1} = -(x + 1)$ and the limit as $x \to 1$ of the last expression is -2.
b) Does $\lim_{x \to 1} F(x)$ exist?
Answer No. The values of the one-sided limits computed above do not agree.
c) Sketch the graph of F.
Answer



Comment This graph is a bit subtle. If $x < -1$ then $x^2 - 1$ is positive, so $F(x) = \dfrac{|x^2 - 1|}{x - 1} = \dfrac{x^2 - 1}{x - 1} = x + 1$ in this region. The graph is thus *three* line segments since $|x^2 - 1|$ is $x^2 - 1$ for $x \leq -1$ and $x \geq 1$ but is $1 - x^2$ for $-1 < x < 1$. A calculator will draw an acceptable graph, except possibly for how the discontinuity is displayed.