Fall, 1995

Math 151 & 153 final exam review problems

The final exam will cover the material in the text up to and including section 5.4 (on the Fundamental Theorem of Calculus). Please be sure to include in your review for the final exam any previous suggested review problems, and, especially, any previous exams. If we thought it was important before, we'll probably still believe it is important now! Note also that the problems below are in *no* particular order, and are intended *only* to be representative of the general type of problem which may be asked on the final.

1. a) Write the definition of derivative as a limit and use this definition to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.

b) Use your answer to a) to write an equation for the line tangent to the curve $y = \dfrac{1}{\sqrt{x}}$ at the point on the curve where $x = 4$.

c) Sketch the graph of the curve named in b) and of the line found in b) on the same coordinate axes.

2. Suppose $F(x) = x^3 - x^2 - 8x + 1$ on the domain $[-2, 2]$. Use calculus to locate precisely the absolute maximum and absolute minimum of F on this domain. Similarly, find any local maxima and minima of F on this domain.

3. An ice cube is melting so that its volume is steadily decreasing at .03 cc/minute. At a certain time, the length of an edge of the ice cube is 2 cm. At that time, how fast (in cm^2/min) is the area of the ice cube changing? Is it decreasing or increasing?

4. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, where should an object be placed on the line between the sources so as to receive the least illumination?

5. a) Suppose $G(x) = x \ln x - x$. Compute $G'(x)$.

b) Use your answer to a) to evaluate $\displaystyle\int_1^{e^2} \ln x \, dx$ as completely as you can in terms of constants familiar to you from calculus.

6. At what point on the curve $y = \dfrac{1}{\sqrt{x}}$ does the triangle formed by the $x-$axis, the $y-$axis, and the tangent to the curve have area 9?

7. If $f(x^3 + x + 1) = x^3$, find $f'(1)$ and $f'(3)$.

8. A function $y = f(x)$ is defined implicitly by

$$y^3 + 3y + x^2 = 4 \, .$$

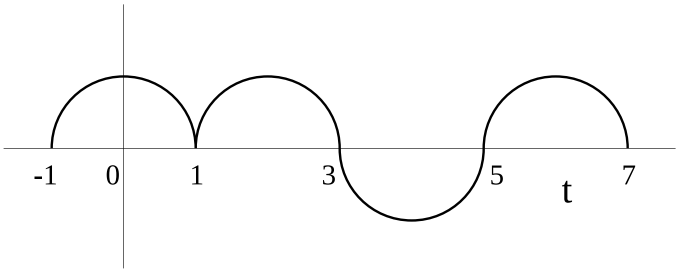
a) Find $\dfrac{dy}{dx}$ in terms of x and y.

b) Find the equation of the tangent line to the curve described by the equation above at the point $(x, y) = (2, 0)$.

c) Show that the function $y = f(x)$ has a maximum value for all x, and find that maximum.

9. What is $\displaystyle\lim_{x \to 0} \frac{2^{x \sin x} - 1}{x^2}$?

10. Below is a graph of $y = f(t)$.



Graph of $y = f(t)$

Note that each segment of the graph is a semicircle.

Suppose that $g(x) = \int_{-1}^{x} f(t)\, dt$.

a) What are $g(-1)$ and $g(0)$ and $g(5)$?

b) In what intervals is g increasing? In what intervals is g decreasing?

c) Sketch a graph of $y = g(x)$ for $-1 \le x \le 7$.

11. Suppose that Q is an increasing differentiable function whose domain is the interval $[1, 5]$, and that Q and its derivative have the following values:

x	$Q(x)$	$Q'(x)$
1	2	0
2	3	1
3	7	3
4	7.5	5
5	8	0

If $W(x) = Q(x^2)$, what are the domain and range of W? What are $W(2)$ and $W'(2)$? If R is the function inverse to Q and S is the function inverse to W, write a formula for S in terms of R and algebraic functions.

12. The polynomial

$$P(x) = x^4 + 3x^3 + x^2 - 1$$

has ____ real roots*. Fill in the blank. You may first choose to graph this function approximately, but any assertion you make should then be verified using calculus, with suitable explanations. You *may* find it helpful to investigate extrema of $y = P(x)$, etc.

13. Let f be defined by

$$f(x) = \begin{cases} -x^2, & \text{if } x \le 0 \\ 2x^2, & \text{if } 0 < x < 1 \\ 4x - 3, & \text{if } 1 \le x \end{cases}.$$

a) Is f defined at 1? If so, what is $f(1)$?

b) Is f continuous at 1? Explain.

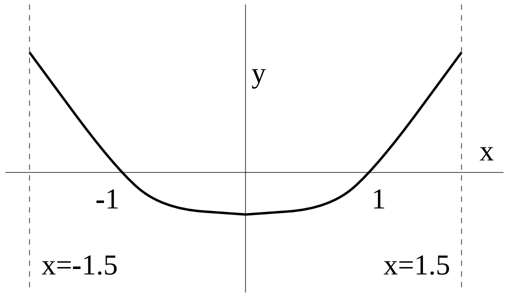
c) What are $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$ and $\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$?

d) Is f differentiable at 0? Explain.

e) Is f differentiable at 1? Explain.

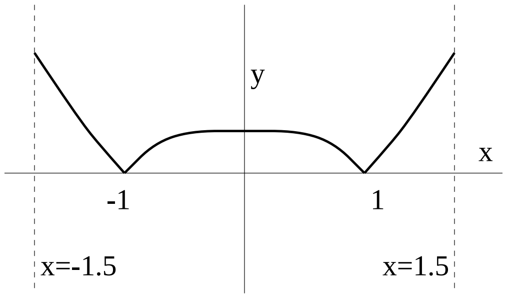
* This may be either 0 or 1 or 2 or 3 or 4.

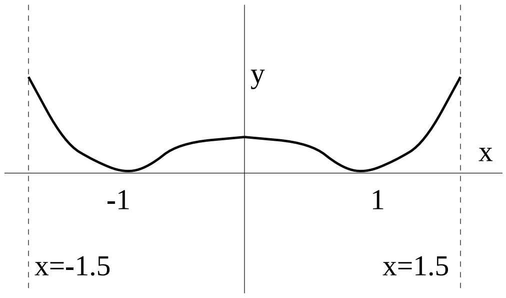
14. Suppose $y = Q(x)$ is the graph given below:



Graph of $y = Q(x)$

a) One of the graphs below is that of $y = |Q(x)|$ and the other one is $y = \big(Q(x)\big)^2$. Which one is which? Why?





b) What are the domains of $y = \sqrt{Q(x)}$ and $y = \dfrac{1}{\sqrt{Q(x)}}$?

15. Suppose $B(w) = 3\cos(5w + \sin w)$. Explain why the inequality

$$|B(w_1) - B(w_2)| \leq 18|w_1 - w_2|$$

is true for *any* numbers w_1 and w_2.

16. What's the highest the curve $y = K(x)$ can be if $K''(x)$ is -9 always and if also $K(0) = 0$ and $K(2) = 0$?

17. Suppose you know that

$$M(2) = 3 \text{ and } M'(x) = \frac{1}{1 + x^3}\ .$$

a) If w is a small number, write an approximation (the linearization of M at 2) for $M(2 + w)$.

b) Is your answer in a) likely to be an underestimate or an overestimate of the true value of $M(2 + w)$? Explain. (Hint: you can easily compute $M''(2)$ and then draw a small piece of the graph of M near 2 and compare it with the tangent line to M when $x = 2$.)

18. Consider the function J whose *derivative* is given by the following formula:

$$J'(x) = \frac{x^2 + 2x}{(x+1)^2(x^2+1)}$$

Answer the following questions as well as you can (you may find it helpful to refer to features of the graph of $y = J'(x)$ in what follows):

a) In what intervals is the function J increasing? In what intervals is the function J decreasing? Why?

b) Does J have any critical numbers? Where does J have local extrema, and what kind of extrema are they?

c) Where are J's points of inflection located? What are the largest intervals in which J is concave up? What are the largest intervals in which J is concave down?