Calculus 151 Problems, Week 3

1. Let

$$f(x) = \begin{cases} 3x-2, & \text{if } x<0 \\ ax+b, & \text{if } 0 \le x \le 1 \\ 3x+4, & \text{if } x>1 \end{cases}.$$

(a) Suppose $a = 2$ and $b = -3$. Graph $f(x)$ for $-2 \le x \le 2$. Find the left and right hand limits of $f(x)$ as x approaches 0 and as x approaches 1.

(b) Find a and b so that the graph of $f(x)$ doesn't have any jumps.

2. Let $f(x) = x^5 - 2x^4 - x + 1 + .05\sin(100x)$.

(a) Prove *without drawing a graph* that the equation $f(x) = 0$ has a solution x in the interval $(0, 1)$. (*Hint:* Use the intermediate value theorem.)

(b) Now plot the graph of $y = f(x)$ on the interval (0,1) (remember to have your calculator in RADIAN mode). Can you determine from your graph how many solutions the equation $f(x) = 0$ has in the interval (0,1)?

(c) Use the ZOOM box technique to find the *smallest* solution $x > 0$ to $f(x) = 0$ with two decimal place accuracy. Show the graph in enough detail to justify your answer.

3. Let $f(x) = 3x - 1$. Then $\lim_{x\to 1} f(x) = 2$ (what limit laws allow you to make this conclusion?). In this problem you will prove this limit statement using the ϵ-δ definition of a limit. Use the graph of $y = f(x)$ to answer (a) and (b), and then use some algebra to answer (c) (check your answer to (c) by the special cases (a) and (b)).

a) Find a number $\delta_1 > 0$ such that $|f(x) - 2| < 0.1$ whenever $|x - 1| < \delta_1$.

b) Find a number $\delta_2 > 0$ such that $|f(x) - 2| < 0.01$ whenever $|x - 1| < \delta_2$.

c) Let $\varepsilon > 0$ be given. In terms of ε, find a number $\delta > 0$ such that $|f(x) - 2| < \varepsilon$ whenever $0 < |x - 1| < \delta$.

4. In many interesting cases, you cannot find a limit just by 'plugging in' a number. Here are some examples that we will study in more detail later. For now, just use your calculator as an experimental tool to guess the limiting behavior as $x \to 0$ of these functions with domain indicated:

a) $Y_1(x) = (1+x)^{1/x}$, with domain $(-1, 0)$ and $(0, +\infty)$.

b) $Y_2(x) = \left(1+x^2\right)^{1/x}$, with domain $(-\infty, 0)$ and $(0, +\infty)$.

c) $Y_3(x) = \left(1+\sqrt{|x|}\right)^{1/x}$, with domain $(-\infty, 0)$ and $(0, +\infty)$.

(You may want to consider behavior as $x \to 0$ from each side, especially for Y_3.)

5. a) The function S (the "squaring function") has domain all of $\mathbb{R}$ (all real numbers) and its values are given by $S(x) = x^2$ for all x. Now consider the function T whose domain is also all of $\mathbb{R}$ and which is defined by

$$T(x) = \begin{cases} S(x) & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$

Sketch a graph of T. What is $\lim_{x \to 5} T(x)$ and what is $\lim_{x \to 3} T(x)$? Prove your answer using the $\epsilon - \delta$ definition of "limit" and the limit laws.

b) Suppose S is again the squaring function defined above. Now an evil interstellar visitor comes and *changes* exactly *one million values* of S and thus creates a new function, V. What can you say about $\lim_{x \to a} V(x)$ for all values of a? Give reasoning to support your assertions.