Calculus 151 Problems, Week 5

1. Suppose $f(x)$ is a differentiable function. Find expressions for the derivatives of the following functions in terms of $f(x)$ and $f'(x)$:

(a) $x^2 f(x)$

(b) $\dfrac{f(x)}{x^2}$

(c) $\dfrac{x^2}{f(x)}$

(d) $\dfrac{1 + xf(x)}{\sqrt{x}}$

2. Define

$$f(x) = \begin{cases} x^2 + p, & \text{if } x \geq 0 \\ 4 - x^2, & \text{if } x < 0 \end{cases} \qquad g(x) = \begin{cases} px^2 + x, & \text{if } x \geq 0 \\ x^3 + p^2 x, & \text{if } x < 0 \end{cases}$$

a) Find all values of p for which $f(x)$ is continuous everywhere, and all values of p for which $g(x)$ is continuous everywhere.

b) Now find all values of p (if there are any) so that $f(x)$ is differentiable everywhere, and all values of p (if there are any) so that $g(x)$ is differentiable everywhere. Draw the graphs of $y = f(x)$ and $y = g(x)$ for these values of p.

3. a) Find $\displaystyle\lim_{x \to 0} \frac{3\sin(8x)}{9x}$ and $\displaystyle\lim_{x \to 0} \frac{x}{\tan(9x)}$. Prove your answers using limit theorems.

b) Now find $\displaystyle\lim_{x \to 0} \frac{x + 3\sin(8x)}{\tan(9x)}$. Prove your answer using limit theorems.

4. Define

$$f(x) = \begin{cases} x^{-1} \sin x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

a) Graph $y = f(x)$ in the range $-2\pi \leq x \leq 2\pi$ and prove (by limit results) that $f(x)$ is continuous for all x.

b) Find a formula for the derivative $f'(x)$ when $x \neq 0$, and plot its graph in the same window as in a). Determine from your graphs the range of x values for which the tangent lines to $y = f(x)$ have negative slope.

c) Does $\lim_{x \to \infty} f(x)$ exist? Prove your answer. (*Hint:* use the Squeeze Theorem.)