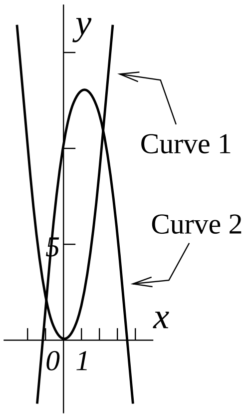
640:152:09-11 **Answers to version A of Exam #1** **2/28/97**

(12) 1. a) Sketch the region which is enclosed by the curves $y = x^2$ and $y = -x^2 + 4x + 6$. Label the curves.



Curve 1 is $y = x^2$ and Curve 2 is $y = -x^2 + 4x + 6$.

b) Set up and compute a definite integral for the area of the region sketched in a). Compute this integral using calculus.

Answer We find the area by integrating `Top` − `Bottom` from `Left` to `Right`. `Top` and `Bottom` are clear, and to get the others, we need to find the intersections of the curves. So $x^2 = -x^2 + 4x + 6$ leads to $2x^2 - 4x - 6 = 0$ leads to $x^2 - 2x - 3 = 0$ leads to $(x-3)(x+1) = 0$ and you are back in the familiar wonderland of lower-level math exams: expressions can be factored easily, solutions are small integers, etc. The area desired is therefore $\int_{-1}^{3} (-x^2 + 4x + 6) - x^2 dx = \int_{-1}^{3} -2x^2 + 4x + 6 dx = -\frac{2}{3}x^3 + 2x^2 + 6x \Big]_{-1}^{3}$

Then the answer is gotten by stuffing in numbers. If you do more "simplification" than is required or wanted, $\frac{64}{3}$ should appear as the answer.

c) Set up a definite integral for the volume obtained when the region sketched in a) is revolved about the x−axis. Describe the method you are using. Do **not** evaluate the integral.
Answer Use the method of washers. The volume desired will be $\pi(\texttt{Outer radius})^2 - \pi(\texttt{Inner radius})^2$ integrated again from `Left` to `Right`. So it is $\int_{-1}^{3} \pi(-x^2 + 4x + 6)^2 - \pi(x^2)^2 dx$.

(10) 2. Use integration by parts to verify that

$$\int_0^1 xe^{3x}\, dx = \frac{2}{9}e^3 + \frac{1}{9}$$

Answer Here we go. I'll get the antiderivative first and then evaluate.

$$\begin{array}{ccccc} \int xe^{3x}\, dx & = x\frac{1}{3}e^{3x} & - & \int e^{3x}\, dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C \\ \scriptstyle\int u\, dv & \scriptstyle = uv & \scriptstyle - & \scriptstyle\int v\, du \end{array}$$

$$\left.\begin{array}{r} u = x \\ dv = e^{3x}\, dx \end{array}\right\} \left\{\begin{array}{l} du = dx \\ v = \frac{1}{3}e^{3x} \end{array}\right.$$

We get the answer desired by plugging in: $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\Big]_0^1 = \left(\frac{e^3}{3} - \frac{1}{9}e^3\right) - \left(-\frac{1}{9}\right) = \frac{2}{9}e^3 + \frac{1}{9}$.

(18) 3. I've been told by a silicon friend that

$$\int_0^\infty \frac{1}{(x+1)(x^2+1)}\, dx = \frac{1}{4}\pi$$

Check that this is correct using calculus.
Comment: you should first find an antiderivative of the integrand, and then carefully evaluate the improper integral.
Answer This is an involved computation. First we'll need partial fractions to get the antiderivative. Write $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)}{(x+1)(x^2+1)}$, so that $1 = A(x^2+1) + (Bx+C)(x+1)$. When $x = -1$ we see that $A = \frac{1}{2}$. Comparing x^2 coefficients of both sides, we see that $B = -1/2$. Finally, comparing constant coefficients on both sides, we see that $1 = A + C$ so $C = \frac{1}{2}$.

Now compute: $\int \frac{1}{(x+1)(x^2+1)}\, dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1}\, dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\, dx =$

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$\frac{1}{2}\ln(x+1)+-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan(x)+C$. We need to apply this antiderivative computation to the definite integral in the problem. $\int_0^\infty$ has to be evaluated as $\lim_{W\to\infty}\int_0^W$ (the letter W has been picked randomly here). So we need to consider
$\frac{1}{2}\ln(x+1)+-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan(x)\Big]_0^W = \frac{1}{2}\ln(W+1)+-\frac{1}{4}\ln(W^2+1)+\frac{1}{2}\arctan(W)$
(at least all the terms with $x=0$ are 0, making the manipulations a little bit easier) and see what happens as $W\to\infty$. Look carefully at the pieces. The values of log can be combined:
$\frac{1}{2}\ln(W+1)+-\frac{1}{4}\ln(W^2+1)=\ln\left(\frac{\sqrt{W+1}}{(W^2+1)^{1/4}}\right)$ — a monstrous expression if we needed to evaluate it when $W=17.486$ but we only want its asymptotic behavior for large W. The dominant powers on the top and the bottom of this expression are the same: just $W^{1/2}$, and therefore, as $W\to\infty$, the expression inside the log goes to 1 and the log of it goes to 0. Wow! Now all that's left to consider is $\frac{1}{2}\arctan(W)$. As $W\to\infty$, $\arctan(W)\to\frac{\pi}{2}$. So, finally, the improper integral converges, and its value is indeed $\frac{\pi}{4}$.

(16) 4. Compute the following integrals by finding antiderivatives and using the Fundamental Theorem of Calculus.

a) $\int_0^{\pi/2}(\sin x)^3\,dx$ **Answer** If $u=\cos x$ so $du=-\sin x\,dx$ and $(\sin x)^2=1-(\cos x)^2=1-u^2$, then $\int(\sin x)^3\,dx=\int(\sin x)^2\,\sin x\,dx=-\int 1-u^2\,du=-u+\frac{1}{3}u^3+C=-\cos x+\frac{1}{3}(\cos x)^3+C$. And evaluating: $\int_0^{\pi/2}(\sin x)^3\,dx=-\cos x-\frac{1}{3}(\cos x)^3\Big]_0^{\pi/2}=-(-1+\frac{1}{3})=\frac{2}{3}$. (0 from $x=\frac{\pi}{2}$ since $\cos(\frac{\pi}{2})=0$.)

b) $\int_0^1\frac{1}{1+\sqrt{x}}\,dx$ **Answer** One possible substitution is a bit tricky: $u=\sqrt{x}$ so $u^2=dx$ and $2u\,du=dx$. There's a little bit of backwards reasoning here. $\int\frac{1}{1+\sqrt{x}}\,dx=\int\frac{1}{1+u}2u\,du=\int\frac{2u}{1+u}\,du=\int 2-\frac{2}{1+u}\,du$ $=2u-2\ln(1+u)+C=2\sqrt{x}-2\ln(1+\sqrt{x})+C$ Remember to divide $2u$ by $1+u$ – the first step always when integrating rational functions is to get a proper fraction, with {degree of `Top`} < {degree of `Bottom`}.
So $\int_0^1\frac{1}{1+\sqrt{x}}\,dx=2\sqrt{x}-2\ln(1+\sqrt{x})\Big]_0^1=2-2\ln 2$. The $x=0$ terms have 0 contribution.
Comment: I usually check my computations these days using some sort of "silicon friend" – these days mostly `Maple`. `Maple` astonished me by giving the answer $2-\ln(-2)$ for this definite integral. I have been unable to figure out what's broken. Two other symbolic manipulation packages I checked (`Macsyma`, quite ancient, and `Derive`, a current commercial product) reported the correct answer.

(12) 5. a) Suppose m and n are positive integers. Find a reduction formula for

$$\int x^m(\ln x)^n\,dx$$

(Here the object is to reduce n, since if we can push n to 0 we'll just have a polynomial to integrate, which is easy.)

Answer Again, integrate by parts.

$$\underbrace{\int x^m(\ln x)^n\,dx}_{\int u\,dv} \underbrace{=}_{=} \underbrace{\left(\frac{x^{m+1}}{m+1}\right)\cdot(\ln x)^n}_{uv} \underbrace{-}_{-} \underbrace{\int n(\ln x)^{n-1}\cdot\left(\frac{1}{x}\right)\left(\frac{x^{m+1}}{m+1}\right)dx}_{\int v\,du}+C$$

$$\left.\begin{array}{r} u=(\ln x)^n \\ dv=x^m\,dx\end{array}\right\}\left\{\begin{array}{l} du=n(\ln x)^{n-1}\left(\frac{1}{x}\right)dx \\ v=\frac{x^{m+1}}{m+1}\end{array}\right.$$

This is *all* I wanted: perhaps the question could have been written more clearly. I will clean up a bit since this formula will be used twice in part b):

$$\int x^m (\ln x)^n dx = \left(\frac{x^{m+1}}{m+1}\right)(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx + C$$

b) Use the formula obtained in a) to compute $\int x^{20} (\ln x)^2 \, dx$.

Answer Since $n = 2$ we'll need two applications of the formula above.

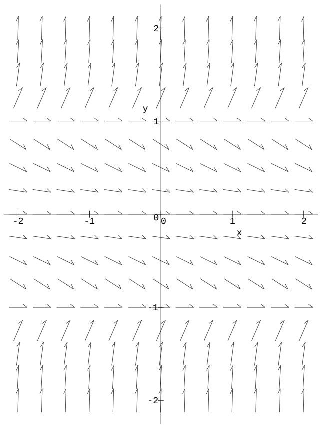
$$\int x^{20} (\ln x)^2 \; dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\int x^{20} \ln x \; dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20} \; dx\right) =$$

$$\frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20} \; dx\right) = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\cdot\frac{x^{21}}{21}\right) + C$$

I doubt whether I would "simplify" anything here unless forced to.

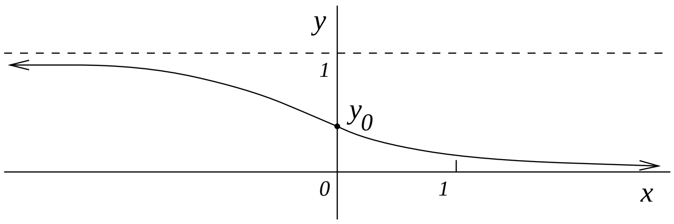
(12) 6. Below is a sketch of the direction field for the differential equation

$$y' = 2y^2(y+1)(y-1)$$



Comment on the **Answer**: I would just draw along with the direction field, making sure that my solution curves were tangent to the little pieces of straight lines whenever they touched.

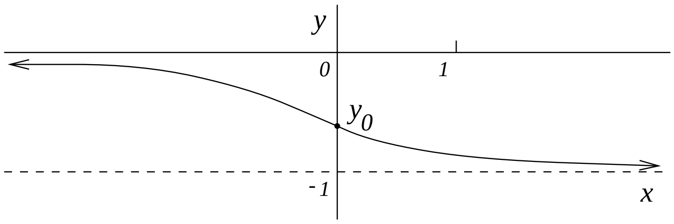
a) Sketch a solution of this equation with initial condition $y(0) = y_0$ when $0 < y_0 < 1$. What are $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$ for such solutions?



$\lim_{x\to\infty} y(x) = 0$

$\lim_{x\to-\infty} y(x) = 1$

b) Sketch a solution of this equation with initial condition $y(0) = y_0$ when $-1 < y_0 < 0$. What are $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$ for such solutions?



$\lim_{x\to\infty} y(x) = -1$

$\lim_{x\to-\infty} y(x) = 0$

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(10) 7. Solve the differential equation $y' = 3 - 2y$ subject to the initial condition $y(0) = 5$. (Please write y explicitly as a function of x.)

Answer This is a separable equation. $\frac{dy}{dx} = 3 - 2y$ leads to $\frac{dy}{3-2y} = dx$, integrating to $-\frac{1}{2}\ln|3-2y| = x + C$. I'd probably plug in the initial conditions here, getting $-\frac{1}{2}\ln|3-10| = 0 + C$ so $C = -\frac{1}{2}\ln 7$. Then $\ln|3-2y| = (\ln 7) - 2x$ so $|3-2y| = e^{(\ln 7)-2x} = e^{\ln 7}e^{-2x} = 7e^{-2x}$, and $3 - 2y = \pm 7e^{-2x}$ so $y = \frac{3}{2} \pm \frac{7}{2}e^{-2x}$, and I would choose the "+" sign so the initial condition would be fulfilled: $y = \frac{3}{2} + \frac{7}{2}e^{-2x}$. There are certainly alternative algebraic forms of the answer that are acceptable.

(12) 8. `Maple` is *not* able to find an antiderivative of $\left(1+x^3\right)^{5/2}$. It does report the numerical approximation

$$\int_0^1 \left(1+x^3\right)^{5/2}\,dx \approx 1.92$$

Here you will use the Trapezoid Rule to get this approximation.

a) First, find n (the number of subdivisions) so that you will be sure that the Trapezoidal Rule estimate with n subdivisions will be within 10^{-2} of the true value of the definite integral. (You may use the error bound $\frac{K(b-a)^3}{12n^2}$ where K is an overestimate of the magnitude of the second derivative.)

Answer We need to get K. To do this, we need the second derivative of $g(x) = \left(1+x^3\right)^{5/2}$. So: $g'(x) = \frac{5}{2}\left(1+x^3\right)^{3/2}3x^2$ and $g''(x) = \frac{5}{2}\cdot\frac{3}{2}\left(1+x^3\right)^{1/2}\left(3x^2\right)^2 + \frac{5}{2}\left(1+x^3\right)^{3/2}6x$.
All of the "pieces" of g'' are positive and increasing functions of x, so that we can get an overestimate by looking at $g''(1)$. You *must* write some reason for this overestimate, since it is certainly *not* true that any function's value at 1 dominates its value everywhere on $[0,1]$.
But $g''(1) = \frac{15}{4}\sqrt{2}\cdot 9 + \frac{5}{2}\sqrt{2}\cdot 6 = \frac{195}{4}\sqrt{2} \leq 70$. It is certainly not necessary to do the involved arithmetic to get the "195", but a computed <u>over</u>estimate should be gotten. If you are following this approach, the estimate $g''(1) \leq 100$ would be valid, but an estimate of 50 would not.
With $K = 70$ we need to find n, knowing that $a = 0$ and $b = 1$. So we consider the inequality

$$\frac{70(1-0)^3}{12n^2} < 10^{-2} = \frac{1}{100}$$

I get $n > 24.15$ so I can take n to be 25. Any integer greater than 24 will be valid according to this error analysis. If you choose $n = 10{,}000$ you're going to have some delay waiting for an answer to b) on most hand-held graphing calculators! (See the **Note** below.)
b) Write down the sum resulting from applying the Trapezoid Rule which you assert is within $\pm.01$ of the integral's value. Then use your calculator to actually compute this Trapezoid Rule approximation to **5** decimal places and report this result.
Answer One algebraic form of the sum for the Trapezoid Rule on the interval $[a,b]$ with n subdivisions is

$$\triangle x\left(\frac{g(a)+g(b)}{2} + \sum_{j=1}^{n-1} g(a + j\triangle x)\right)$$

where $\triangle x = \frac{b-a}{n}$. Part a) allows us to use $n = 25$. Of course, $b - a = 1 - 0 = 1$ and $g(x) = \left(1+x^3\right)^{5/2}$. The specific sum* we need here is therefore $\frac{1}{25}\left(\frac{g(0)+g(1)}{2} + \sum_{j=1}^{24} g\left(0 + j\left(\frac{1}{25}\right)\right)\right)$. The Trapezoid Rule program on my TI-82 reports the answer 1.92448 for this sum. The `fnInt` program under the `MATH` key reports the value 1.921652852 for the same integral, using a much more sophisticated algorithm ("adaptive Gaussian quadrature"). `Maple` reports the same answer as `fnInt`.

Note Using my perhaps inaccurate timing, the TI-82 took 2 seconds with $n = 10$, 12 seconds with $n = 100$, and 119 seconds with $n = 1{,}000$. The value reported when $n = 1{,}000$ was 1.921654619. I did not try to time $n = 10{,}000$ — I hope you can see why.

* Summation notation is precise, helps the lazy, and saves time, too! Learn to use it, please.