640:152:09-11 **Answers to version A of Exam #2** **4/18/97**

(14) 1. Consider the power series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\, x^n$.

a) For which values of x does this series converge absolutely?

Answer Let's use the ratio test. Here $a_n = \frac{x^n}{\sqrt{n}}$ so (after the usual algebra!) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n}}{\sqrt{n+1}}\,|x|$. As $n \to \infty$ this ratio $\to |x|$. So if $|x| < 1$ the series converges absolutely, and if $|x| > 1$ the series diverges. The cases $x = \pm 1$ must be handled separately. Before we do that, I should mention that the root test also applies easily here, since $\left|a_n\right|^{1/n} = (n^{-1/2})^{1/n}|x| = (n^{1/n})^{-1/2}|x|$. If you remember that $n^{1/n} \to 1$ as $n \to \infty$ then we get the same information that the ratio test gave.

Now to handle the cases $x = \pm 1$. When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$, and it therefore diverges. When $x = -1$ the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, and this converges (but not absolutely!) using the alternating series test.

So the answer is: the series converges absolutely for $-1 < x < 1$.

b) For which values of x does this series converge conditionally?

Answer $x = -1$ as described above.

c) What is the radius of convergence of this series?

Answer 1

(14) 2. Use Taylor's Theorem to explain how to compute $e^{1/3}$ to an accuracy of $\pm.0001$; you should write down a specific finite sum which is the desired approximation, and be sure to show that the error is within the desired tolerance. You are *not* asked to do the actual computation.

Answer $e^x = T_n(x) + R_n(x)$ where (in this case) $T_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$ and $R_n(x) = e^z \frac{x^{n+1}}{(n+1)!}$ for some z between x and 0. Here $x = \frac{1}{3}$ so (since $0 < z < \frac{1}{3} < 1$ and e^z is increasing and $e < 3$) we know $\left| R_n\left(\frac{1}{3}\right) \right| < 3\cdot\frac{1}{3^{n+1}}\cdot\frac{1}{(n+1)!}$. We need an n so that the last group of terms is less than $.0001 = 10^{-4} = \frac{1}{10{,}000}$. $n = 5$ does this since $3^5 = 243$ and $6! = 720$. By the way, so does $n = 15$ and $n = 10^{100}$. We then know that $e^{1/3} = \sum_{j=0}^{5} \frac{\left(\frac{1}{3}\right)^j}{j!} \pm .0001$. By the way, `Maple` reports that this approximation is 1.395610425 and that the "true value" is (approximately!) 1.395612425, so I guess we're right.

(14) 3. a) Explain why the series $F(x) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}x)}{6^n}$ converges for all values of x.

Answer Since we know that absolutely convergent series must converge, we can establish the convergence of the series for $F(x)$ by considering the series $\sum_{n=1}^{\infty} \frac{|\sin(\sqrt{n}x)|}{6^n}$. But $|\sin(\text{anything})| \le 1$. And we can compare series with non-negative terms to establish convergence. Therefore, since we know $\sum_{n=1}^{\infty} \frac{|\sin(\sqrt{n}x)|}{6^n} \le \sum_{n=1}^{\infty} \frac{1}{6^n}$. The series written last is a geometric series with ratio$= \frac{1}{6} < 1$, so it must converge. The series for F converges absolutely and so converges.

b) Suppose $S_N(x) = \sum_{n=1}^{N} \frac{\sin(\sqrt{n}x)}{6^n}$. Find N so that $|F(x) - S_N(x)| < .001$ for all x. Explain why your assertion about this N is correct.

Answer The difference between $F(x)$ and $S_N(x)$ is an "infinite tail": $T_N(x) = \sum_{n=N+1}^{\infty} \frac{\sin(\sqrt{n}x)}{6^n}$ whose absolute value we can estimate (using the $\triangle \leq$) as follows: $|T_n(x)| \leq \sum_{n=N+1}^{\infty} \frac{|\sin(\sqrt{n}x)|}{6^n} \leq \sum_{n=N+1}^{\infty} \frac{1}{6^n}$. There's an explicit formula for the sum of a convergent geometric series: the sum is $\frac{a}{1-r}$. Here a (the first term) is $\frac{1}{6^{N+1}}$ and the ratio, r, is $\frac{1}{6}$. So the sum is $\frac{\frac{1}{6^{N+1}}}{1-\frac{1}{6}} = \frac{1}{5}\cdot\frac{1}{6^N}$. We'll be done if we can choose N so that this is $< .001 = \frac{1}{1{,}000}$. But $5 \cdot 6^3 = 1{,}080$ so $N = 3$ is good enough (so is $N = 7$ and $N = 308$ but giving 308 as your answer would make writing out the response to part c) extremely tedious!).

c) Use your answer to b) to write out the specific sum (not using any $\sum$'s) which will approximate $F(1.23)$ within an error of $\pm.001$. You are *not* asked to evaluate this sum.

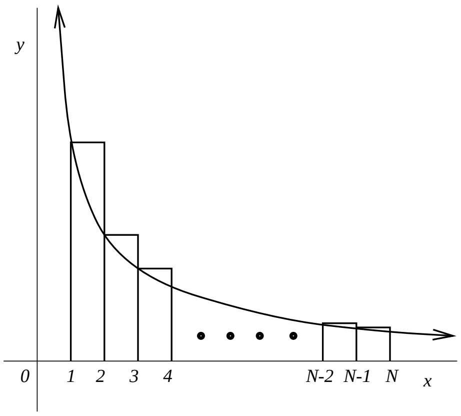
Answer $\frac{\sin\left(\left(\sqrt{1}\right)1.23\right)}{6^1} + \frac{\sin\left(\left(\sqrt{2}\right)1.23\right)}{6^2} + \frac{\sin\left(\left(\sqrt{3}\right)1.23\right)}{6^3}$

(12) 4. a) Explain why the infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Answer One "scholarly" answer: this series is a p–series with $p = \frac{1}{2} < 1$ and therefore it diverges. This answer is certainly acceptable. Another acceptable answer would apply the Integral Test: analyze the improper integral $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ and show that it diverges, which is not too hard. Even if you *don't* explicitly use the Integral Test here, you'll need the logic underlying it to do b).

b) If $S_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}}$, find a specific N so that $S_N > 1{,}000$. Explain why your assertion about this N is correct.

Answer I'll draw the usual picture (the curve shown in the picture to the right is supposed to be $y = \frac{1}{\sqrt{x}}$). Then $S_N > \int_1^N \frac{1}{\sqrt{x}}\,dx$ and all we need to do is evaluate the integral (which is easy) and then find an N making the integral at least 1,000. The integral underestimates the sum, and we will be done. So: $\int_1^N \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_1^N = 2\sqrt{N} - 2$. Let's see: $2\sqrt{N} - 2 \geq 1{,}000$ when $N \geq 251{,}001$, so we could take $N = 251{,}001$. Or, for that matter, $N = 10^{100}$ would also work because it is larger than $251{,}001$. But I think my colleagues want something a bit smaller, as I have explained in class.



Comment I did ask `Maple` to compute a partial sum of this series. I first asked it to compute the sum when $N = 10{,}000$. `Maple` reported that the sum was 198.5446427 but it took more than 35 seconds to do it on a moderately fast machine, using up a lot of memory. So I wrote a short program in Pascal (an antique language) and (further evidence that compiled programs generally run far faster than similar programs in interpreted languages) I was told in less than a second that $S_{251{,}001} = 1{,}000.54064349485$ – of course with this number of arithmetic operations there is the possibility of serious floating-point error. I don't think I'd trust the "exact" value too much!

(18) 5. In this problem $V(x) = (1+x)^{2/5}$.

a) If x is "small", $V(x)$ is approximately $1 + Ax + Bx^2$. What are A and B? (Hint: $T_2(x)$.)

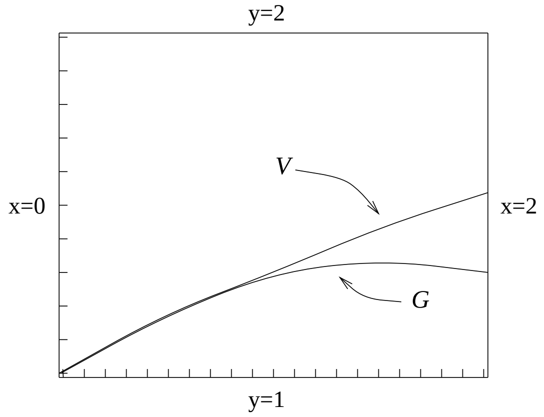
Answer We'll need several derivatives of V. So here we go:

$V'(x) = \left(\frac{2}{5}\right)(1+x)^{-3/5}$ $\quad V''(x) = \left(\frac{2}{5}\right)\left(-\frac{3}{5}\right)(1+x)^{-8/5}$ $\quad V'''(x) = \left(\frac{2}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)(1+x)^{-13/5}$

The third derivative will be needed in part c). Here we just need the first two to create $T_2(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 = 1 + \frac{2}{5}x - \frac{3}{25}x^2$. So $A = \frac{2}{5}$ and $B = -\frac{3}{25}$.

b) Define $G(x)$ to be $1 + Ax + Bx^2$ where A and B are the values found in part a). Sketch a graph on the same coordinate axes of both $V(x)$ and $G(x)$ in the window $0 \le x \le 2$ and $1 \le y \le 2$. Label the curves you sketch. Marks have been drawn on both the vertical and horizontal sides of the window below at intervals of .1.

Answer



c) There are positive numbers P so that if $0 \le x < P$, then $|V(x) - G(x)| < 10^{-10}$. Use Taylor's Theorem to find such a P and be sure to include an explanation of why the P you give is a valid answer.

Answer Taylor's Theorem states that this difference is $R_2(x) = V'''(z)\frac{x^3}{3!}$ where z is some number between 0 and x. I want to **over**estimate the magnitude of $R_2(x)$ by something easier to work with. $|V'''(z)| = \frac{48}{125}(1+z)^{-13/5}$. Since $x > 0$ and $0 < z < x$ the function $(1+z)^{-13/5}$ is decreasing and can be overestimated by its value at 0, which is 1. So a suitable overestimate for $|R_2(x)|$ is $\frac{48}{125} \cdot \frac{1}{6}x^3$. Let me shock you: I will further overestimate this by just x^3. For which x's will this be less than 10^{-10}? Certainly this will be true if $0 \le x < 10^{-10/3}$. Therefore I will take $P = 10^{-10/3}$. Note that there are many valid answers for P and your chief task here is to supply the justification for your choice of P.

(12) 6. True or false? If false, give an example to show that the implication is not true. If true, briefly explain why.

a) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

Answer True. If $\sum_{n=1}^{\infty} a_n$ converges, eventually the terms are less than 1. Then the corresponding terms of the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ will all be *greater* than 1. That series cannot converge since its terms do not approach 0.

b) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

Answer False. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and all of its terms are positive. The series of its reciprocals is $\sum_{n=1}^{\infty} n$ which surely diverges. A colleague suggested a better and simpler example: take all the

a_n's to be 1! b) then asserts that this series converges if it diverges: a rather oxymoronic (self-contradictory) mathematical joke!

(16) 7. a) Use the Taylor series for $\cos x$ to find a Taylor series for $\cos(x^3)$. Your answer should be in summation notation. Indicate the general term as well as you can.

Answer We know $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, so $\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}$.

b) Use your answer to a) to find a Taylor series for an antiderivative of $\cos(x^3)$.

Answer $\int \cos(x^3)\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \int \frac{x^{6n}}{(2n)!}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(6n+1)(2n)!}$. I didn't write "$+C$" because I'm just finding *one* antiderivative, as requested.

c) Use your answer to b) to write an infinite series for $\int_0^1 \cos(x^3)\,dx$.

Answer This is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(6n+1)(2n)!}\Big]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)(2n)!}$.

d) Find a specific partial sum of the series in c) which is within $\pm.001$ of the value of the definite integral. Indicate why your assertion is correct. You are *not* asked to compute this partial sum.

Answer The answer to c) satisfies all the requirements of the alternating series theorem: the terms alternate in sign, they decrease in magnitude, and their limit is 0. Therefore, *in this case*, the difference between the sum of the whole series and any partial sum is less in absolute value than the magnitude of the first omitted term. So we need to find an n with $\frac{1}{(6n+1)(2n)!} < .001 = \frac{1}{1{,}000}$. But when $n = 3$, $(6n+1)(2n)! = 13{,}680$ (when $n = 2$, it is only 312) so we can approximate the whole sum suitably by only the terms up to $n = 2$. The answer desired is $\sum_{n=0}^{2} \frac{(-1)^n}{(6n+1)(2n)!}$. `Maple` reports the "true" value of $\int_0^1 \cos(x^3)\,dx$ to be .9317044406, while some arithmetic on the sum written above (with three terms) gives the sum a value of .9317765568 – certainly within the margin of error the problem requests.

General comment on the exam and its solutions Computations with infinite series can be mysterious. In this course we've tried to present infinite series as a way of constructing approximations of function values. Error estimation has been very important. We've seen various techniques. They've included Taylor's Theorem as in problems 2 and 5 c), definite integrals as in problem 4, and the ratio test and geometric series (combined with the implication that "absolute convergence inplies convergence") as in problems 1 a) and 3, in addition to the simple-minded idea derived from the alternating series test shown in problem 7 d).

As I remarked during our review session, there are several major fallacies to avoid. One is that *any* sum can be estimated by taking a "partial sum" and then using the first omitted term to estimate the error. For example, if $S = 2 + 7 + 5 + 206$ it is *certainly* false that $S = (2 + 7) \pm 5$. Similarly, you *cannot* estimate the sum of an infinite series by simply truncating it (chopping at off) at one point and saying that the error is no more than the first omitted term. The series $\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ certainly converges since it is a p–series with $p > 1$. The first 10 terms add up to 2.926278238 and the 11$^{\text{th}}$ term is .09069136156, but the whole sum is quite a bit larger than $2.9 + .09$ (it is about 1,000 – you can underestimate it with an integral).

The other common fallacy is using comparison techniques where only a one-sided bound has been established: you *can't* conclude from the fact that $a_n \leq b_n$ for all n that $\sum_{n=1}^{\infty} a_n$ *must* inherit convergence from $\sum_{n=1}^{\infty} b_n$. To see that this implication is false, look at two series, one of whose terms is always -1 (and therefore diverges) and the other of whose terms is always 0 (and therefore converges). Comparison techniques are frequently extended to non-positive series by using absolute convergence as in the solution to problem 3 a). But: unless you tell the reader what you are doing, it is difficult to follow your reasoning, to believe what you write, and to give you credit for what you have done.