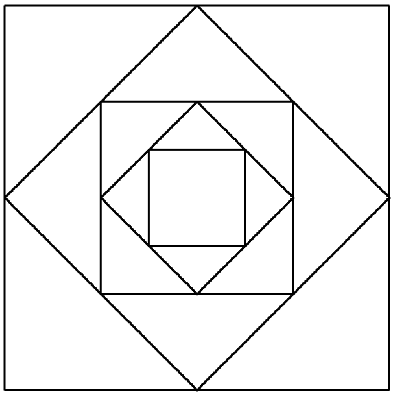
Problems, Week 6

1. An infinite sequence of squares is drawn (the first five are shown), with the midpoints of the sides of one being the vertices of the next. The outermost square has sides which are 1 unit long. What is the <u>sum</u> of the <u>perimeters</u> of all of the squares?



2. Determine whether each of the following sequences converges, and find the limits when they exist. Explain your answers.

a) $a_n = \dfrac{\ln(\ln(n^4))}{(\ln n)^3}$ b) $b_n = \sqrt[n]{6n^8}$

3. Consider the following sequences:

a) $a_n = \left(1 + \left(\frac{1}{n}\right)\right)^n$ b) $b_n = \left(1 + \left(\frac{1}{n^2}\right)\right)^n$ c) $c_n = \left(1 + \left(\frac{1}{\sqrt{n}}\right)\right)^n$

Find the first five terms of each of these sequences. What can you say about the convergence of each of these sequences? (Why?)

4. Suppose α and β are positive numbers. Decide whether the following sequences converge. If they converge, try to find their limits.† (You must give reasons for your assertions!)

a) $d_n = \sqrt[n]{\alpha^n + \beta^n}$ b) $e_n = \sqrt[n]{\alpha^n + \beta}$

5. a) Compute the first five terms of the sequence $\left\{n \arctan\left(\frac{1}{n}\right)\right\}$. Do you think this sequence converges? If you do, what do you think the limit is?

b) Rewrite $n \arctan\left(\frac{1}{n}\right)$ as $\dfrac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}}$. Then use l'Hospital's rule to answer the questions asked in part a).

c) Redo a) and b) for the sequence $\left\{n\left(\sqrt{1 + \frac{3}{n}} - 1\right)\right\}$.

d) Suppose f is a differentiable function and you know that $f(0) = 0$. Then conclude that the sequence $\left\{n\, f\left(\frac{1}{n}\right)\right\}$ converges, and tell what its limit is.

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† Your answers may involve both numbers and their relationship.

6. Consider the differential equation

$$y'' = -y$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$ (this is what is called the "initial velocity" solution). Suppose that the solution looks like

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7 + \dots$$

where the letters $A, B, C, \dots$ are used to represent unknown constants. Use the initial conditions to find values of some of these constants. Then use the differential equation to deduce values of the other constants. Can you see a general pattern? What is the pattern?

Special added bonus question: what well-known function is the "initial velocity" solution? What do you think will be the relation between the series you have discovered and this function?

7. a) Two students are sharing a loaf of bread. Student Alpha eats half of the loaf, then student Beta eats half of what's left, then Alpha eats half of what's left, and so on. How much of the loaf will each student eat?

b) Two students are sharing a loaf of bread. Student Alpha, now hungrier and more ferocious, eats two-thirds of the loaf, then student Beta eats eats half of what's left, then Alpha eats two-thirds of what's left, then Beta eats half of what's left, and so on. How much of the loaf will each student eat?

c) Now let's start with three students: Alpha, Beta, and Gamma. They decide to share a loaf of bread. Alpha eats half of the loaf, passes what's left on to Beta who eats half, and then on to Gamma who eats half, and then back to Alpha who eats half, and so on. How much of the loaf will each student eat?

8. a) If you enter 5 in your 10 digit calculator, and then hit the square root button 20 times, you'll probably get:

$$1.00000\,15348$$

and if you then subtract 1 and multiply by 1,048,576 you'll get:

$$1.60943\,91475$$

but on the other hand, the same calculator will tell you that $\ln 5$ is:

$$1.60943\,79124$$

and I wonder, is this just a coincidence? **Well, is it?** Hint: 1,048,576 is 2^{20}.

b) Given a positive number, x, outline a strategy for computing $\ln x$ only with the "primitive" arithmetic operations $(+,\ \times,\ -,\ /)$ and square root $(\sqrt{})$. Your strategy should involve asserting (and verifying) that a certain sequence which can be easily computed with the listed operations always converges to $\ln x$.

9. Bounded increasing sequences converge.

a) Prove: if $0 \le w \le 87$, then $0 \le \sqrt{3w+42} \le 87$.

b) Prove: if $0 \le u \le v \le 87$, then $0 \le \sqrt{3u+42} \le \sqrt{3v+42} \le 87$.

c) Consider the following recursively defined sequence:

$$\begin{aligned} v_1 &= 1 \\ v_{n+1} &= \sqrt{3v_n + 42} \end{aligned}$$

and compute its first five terms. Prove that this sequence converges.

It sure is interesting that bounded increasing sequences converge. See p.585 of the text.

d) Suppose V is the limit of the sequence defined in c). What is V? (Square the recursive equation and take limits using some limit theorems.)

Note: this process is used in many, many sequences encountered in "real life".

Oh, by the way, bounded increasing sequences converge.