Problems, Week 7

1. Determine which of the following statements are always true. If a statement is false, give an example to show that it fails.

a) If a series doesn't converge, then its n^{th} term doesn't approach 0 as n approaches ∞.

b) If the n^{th} term of a series does not approach 0 as n approaches ∞, then the series diverges.

c) If all of the partial sums of a series are less than some constant L , then the series converges.

d) If a series converges, then there is a constant L such that all of its partial sums are less than L .

2. Determine which of the following series converge and which diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-1}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2^n+1}}{\sqrt{3^n-1}}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{-n}+4^n}{3^n+2^{-n}}$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+n^4}{3^n+n^2}$

3. a) Suppose that $a_n \geq 0$ for all n, and that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ must also converge.
Hint: look at the a_n's: they must get small. *Compare* the size of a small [positive] number and its square.)

b) Give an example to show that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ can converge even though $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge.

4. Consider the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^3+\sqrt{k}}.$$

a) Explain why this sum converges.

b) Find K so that $\displaystyle\sum_{k=K}^{\infty} \frac{1}{k^3+\sqrt{k}} < .0001$.

c) Find $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+\sqrt{k}}$ to within an accuracy of $\pm .0001$.

5. Suppose $z_1 = \dfrac{1}{2}$ and that $z_{n+1} = (z_n)^2$ when $n \geq 1$. Prove either that $\displaystyle\sum_{1}^{\infty} z_n$ diverges, or (if it converges) find its sum to 10 decimal place accuracy.