Problems, Week 8

1. For each fixed value of x, the terms of the series $\sum_{n=1}^{\infty} a_n$ are given by the recursive formula:

$$a_1 = x$$
$$a_{n+1} = \left(\frac{n}{n+1}\right) x^2 a_n$$

Write out the first 5 terms of this series. Guess a general explicit formula for a_n and discover for which values of x this series converges.

2. For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3 x^{4n}}{(3n)!}$ converge? (Be careful: $3n!$ and $(3n)!$ and $3(n+1)!$ and $(3(n+1))!$ are all wonderfully different!)

3. a) How many terms in the following series do you need to add in order to give an answer within $\frac{1}{10}$ of the true value?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

b) We will show later that

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

If you add up the same number of terms of this series as you did in part a), how close will you be to the true value?

5. Conditional convergence is *much much* stranger than you think.

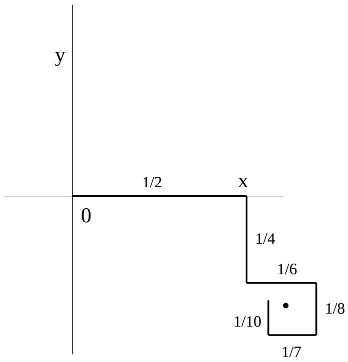
You have a bag containing an infinite number of sticks*. The n^{th} stick has length $\frac{1}{n}$. You're also given a copy of the (x, y)-plane. An alien named Xwyzq will destroy the Earth unless … *you* can win the following

GAME

> Xwyzq will specify a point on the plane, and a positive "tolerance". You will have to select sticks from your bag and lay them out vertically or horizontally, with the end of one stick abutting the beginning of the next, and try to get within the (specified) positive tolerance of the point the alien has selected in the (x, y)-plane. (You lay your first stick starting at the origin.) A small picture of one possible path which gets within .05 of the point $(.6, -.3)$ follows:

---

* Don't bother me with reality while I'm awake, please!



This is a path of sticks whose endpoint is near the point $(.6, -.3)$ (both coordinates of the endpoint are within .05 of what is desired because $\frac{1}{2} + \frac{1}{5} - \frac{1}{7} \approx 0.56$ and because $-\frac{1}{4} - \frac{1}{8} + \frac{1}{10} \approx -0.28$).

Of course the question is:

Can **YOU** prevent the destruction of the Earth?

That is, can you find a finite selection of sticks to get as close as Xwyzq specifies to any point in the plane?

After you've thought about this, you might want to contemplate the answer to the question if the n^{th} stick in the bag had length $\frac{1}{n^2}$. (Yes, this is a weird question, but the phenomenon you're investigating is even weirder.)

6. Consider the series

$$\pm\frac{1}{3} \pm \frac{1}{9} \pm \frac{1}{27} \pm \dots$$

where *you* will choose the signs.

a) Is it possible by clever choice of signs to get 0 as a sum of this series?

b) Is it possible by clever choice of signs to get $\frac{1}{4}$ as a sum of this series?

7.* Suppose $\alpha_n = \frac{1}{n^2}$ except when n is a square, and $\alpha_n = \frac{1}{n^{2/3}}$ when n is a square. The series whose n^{th} term is α_n is thus:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^{3/2}} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^{3/2}} + \dots$$

Does this series converge?

8. For which positive integers k is the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

convergent?

---

* This series occurs in a text first published in 1908 by T.J. I'A. Bromwich, M.A., Sc.D., F.R.S., based on courses of lectures given at Queen's College, Galway. A knowledge of history is valuable for scholars in all fields!