Problems, Week 9

1. Suppose $f(x) = \exp(x^3)$. Find $T_4(x)$ when $a = 0$. (There's got to be a better way!)

2. Find the radius of convergence and the interval of convergence for each of the following power series.

$$\sum_{n=1}^{\infty}(-1)^n n x^{2n} \qquad \sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \qquad \sum_{n=1}^{\infty}\frac{x^n}{(n!)^2} \qquad \sum_{n=1}^{\infty} n! x^n$$

3. Suppose $y = f(x)$ is a solution of the differential equation

$$y' = y + x$$

which satisfies the initial condition $y(0) = 1$.

a) What is $f'(0)$? (Use the differential equation.) What is $f''(0)$? (Differentiate the differential equation and use the information you already have.) What is $f'''(0)$? (Do it again!) Uhhhh .... maybe you could now even tell me what $f^{(46)}(0)$ is.

b) What is the Taylor series for f centered at $a = 0$? (Be a bit careful. The first few terms are anomalous but the others follow a fairly simple pattern.)

c) If you assume that the Taylor series for f centered at $a = 0$ converges to f and that Taylor series are unique, can you identify f using examples you know? Try to verify directly that the function you identified as f is actually a solution of the initial value problem given in part a).

4. Find $T_3(x)$ for the function $f(x) = \sqrt{16+x}$ and $a = 0$. Use $T_3(x)$ to <u>compute</u> an approximate value of $\sqrt{17}$. First decide the value of x you should take to compute $\sqrt{17}$. Then find an explicit formula for the error. Find a very simple overestimate for the error, and <u>compute</u> this overestimate. Finally, compute $\sqrt{17}$ using some method you think is reliable and check that the numbers you've already gotten are consistent with what you have computed.

5. Suppose we want to replace $\sin x$ by $x - \dfrac{x^3}{3!}$ in some interval centered around 0 (that is, an interval like $-A \le x \le A$ with $A > 0$). Find an A so that the error committed doing such a replacement will be no more than $\pm .001$.

Note: There is *no* unique correct answer to this question. The difficulty lies in making a convincing error estimate, an estimate to show that replacing $\sin x$ by this cubic (third degree) polynomial in the interval you've chosen satisfies the desired error tolerance. By the way, it is not accidental that the polynomial just happens to be $T_3(x)$ [or is it $T_4(x)$?] for $\sin x$ when $a = 0$.

6. For parts a), b) and c), find the Taylor series for each function about the indicated point a, and compute the interval of convergence. Then do part d).

a) $f(x) = \dfrac{1}{1-x}$ and $a = 0$ (There is a direct computation and a short cut. Just do the short cut this time.)

b) $g(x) = \dfrac{1}{1+x^2}$ and $a = 0$ (Easy way: use part a).)

c) $h(x) = \arctan x$ and $a = 0$ (Easy way: use b) and notice that $\arctan x = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$.)

d) What is the value of the sixteenth derivative of $\arctan x$ at $x = 0$? What about the value of the seventeenth derivative there?

7. Write (with summation notation) a finite sum (involving only integers and their products and quotients) which will approximate $e^{-.4}$ with an error less than $\pm 10^{-(10^{10})}$.

Note: You are not asked to actually compute the approximation, but you are asked to assert that one exists and to write it. There is *no* unique correct answer to this question. One method of attack might be to write the approximation as $T_n(-.4)$, where $T_n(x)$ is the n^{th} Taylor polynomial for e^x centered at $a = 0$. The difficulty lies in making a convincing error estimate, an estimate to show that your sum (with the n you will specify!) is actually within the desired tolerance of $e^{-.4}$.

8. Consider the function

$$F(x) = \frac{\sin x}{1-x}$$

$F(x)$ has a Taylor series (of the form $\displaystyle\sum_{j=0}^{\infty} \frac{F^{(j)}(0)}{j!} x^j$). Compute the terms of this Taylor series up to (and including) the x^5 term.

Hint: While a direct computation is possible (by differentiating F repeatedly and then evaluating the derivatives), it is *very* unappealing. You should now know series for both $\sin x$ and $\dfrac{1}{1-x}$. Just multiply the series together. Try to do no unnecessary work: you just need to multiply and add terms whose degree is ≤ 5. This should not be very hard.

9. Taylor series centered at 0 may be found for these functions by a long method or by a short method. Describe both methods, and then use the short one.

a) $f(x) = x^3 e^x$

b) $f(x) = \ln(1 + x^2)$

c) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 0, & x = 0 \end{cases}$

10. a) Can you find a power series centered at 0 which converges at $x = 3$ and diverges at $x = -2$? Give an example of one if you can, or give a reason why if you can't.

b) Can you find a power series centered at 0 which converges <u>only</u> at $x = 0$? Give an example of one if you can, or give a reason why if you can't.

c) Can you find a power series centered at 0 which converges <u>only</u> for x's in the interval $-3 \le x < 3$ (read carefully!)? Give an example of one if you can, or give a reason why if you can't.

11. What is

$$\lim_{x \to 0} \frac{\arctan(x^2) - (\arctan x)^2}{x^4}$$

Don't use l'Hôpital's rule four times! Please don't.

12. Use series techniques to compute each of these integrals to an accuracy of $\pm .0001$. Explain *why* your accuracy is achieved by estimating the error.

$$\int_0^1 \sin(x^2)\, dx \qquad\qquad \int_0^{\frac{1}{2}} \frac{dx}{1 - x^3}$$

13. In certain areas of theoretical computer science, the idea of a *GENERATING FUNCTION* is very important. Here's the definition:

F is a <u>*GENERATING FUNCTION*</u> for a sequence $\{a_n\}_{n=0}^{\infty}$ if $F(x) = \sum_{n=0}^{\infty} a_n x^n$.

For example, the exponential function (just e^x) is the generating function for the sequence $\left\{\frac{1}{n!}\right\}_{n=0}^{\infty}$. Recognition of the generating functions of certain obvious sequences is very important in a number of applications. Below is a list of sequences and a list of functions. Can you match the sequences with their generating functions?

Sequences :	Functions :
a) $\{1, 1, 1, \ldots$ (All 1's)$\}$	α) $\frac{1+x}{(1-x)^3}$
b) $\{0, 1, 2, 3, \ldots, n, \ldots\}$	β) $\ln\left(\frac{1}{1-x}\right)$
c) $\{0, 1, 4, 9, \ldots, n^2, \ldots\}$	γ) $\frac{1}{1-2x}$
d) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$	δ) $\int_0^x \frac{\ln\left(\frac{1}{1-t}\right)}{t}\, dt$
e) $\{0, 1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\}$	ϵ) $\frac{1}{1-x}$
f) $\{1, 3, 3, 1,$ (All the rest 0's)$\}$	μ) $\frac{x}{(1-x)^2}$
g) $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$	ν) $(1+x)^3$

Note: Generating functions turn out to be a very convenient and powerful way to describe and manipulate sequences. You'll see them in probability, also.