February 25, 1996

Answers to Exam #1 for Math 152, sections 1 & 2

Here are the questions together with some answers to version **B** of the first exam. There are certainly other methods of getting the answers to some the questions. I hope these solutions will be useful. Please note: I have added occasional **Comments** not really needed for the solutions which might be interesting or helpful.

(10) 1. Use the method of partial fractions to verify that

$$\int_1^2 \frac{x+2}{x(x+3)}\,dx = \frac{1}{3}\ln(5)$$

Answer First get A and B so that $\frac{A}{x} + \frac{B}{x+3} = \frac{x+2}{x(x+3)}$. Combining the fractions and looking only at the tops gives the equation $A(x+3)+Bx = x+2$. My favorite method is now to look at "magic numbers" for x. So $x=0$ tells me that $A = \frac{2}{3}$ and $x=-3$ tells me that $B=\frac{1}{3}$. Now we can integrate:

$$\int_1^2 \frac{x+2}{x(x+3)}\,dx = \int_1^2 \frac{\frac{2}{3}}{x} + \frac{\frac{1}{3}}{x+3}\,dx = \frac{2}{3}\ln(x) + \frac{1}{3}\ln(x+3)\Big]_1^2 =$$

$$\frac{2}{3}\ln(2) + \frac{1}{3}\ln(5) - \frac{2}{3}\ln(1) - \frac{1}{3}\ln(4) = \frac{1}{3}\ln(4) + \frac{1}{3}\ln(5) - 0 - \frac{1}{3}\ln(4) = \frac{1}{3}\ln(5)$$

Of course I used the facts that $\ln(1)=0$ and $2\ln(2)=\ln(4)$.

(12) 2. Use integration by parts to verify that

$$\int_0^{\pi/2} x^2\cos x\,dx = \frac{1}{4}\pi^2 - 2$$

Answer Integrate by parts twice, reducing the degree each time. I'll first just find the antiderivative, and then evaluate the result at the end.

$$\int x^2\cos x\,dx = x^2\sin x - \int (\sin x)\,2x\,dx$$

$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$\left.\begin{array}{r} u = x^2 \\ dv = \cos x\,dx \end{array}\right\}\left\{\begin{array}{l} du = 2x\,dx \\ v = \sin x \end{array}\right.$$

But $-\int (\sin x)\,2x\,dx = -2\int x\sin x\,dx$. We can integrate *that* by parts:

$$\int x\sin x\,dx = x\,(-\cos x) - \int -\cos x\,dx$$

$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

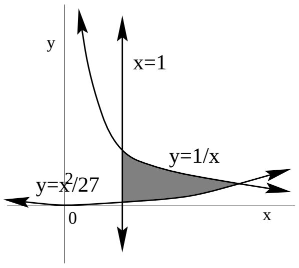
$$\left.\begin{array}{r} u = x \\ dv = \sin x\,dx \end{array}\right\}\left\{\begin{array}{l} du = dx \\ v = -\cos x \end{array}\right.$$

The final integral is easy: $\int -\cos x\,dx = -\int \cos x\,dx = -\sin x + C$.
Now I will try to pack them all together correctly:
$\int x^2\cos x\,dx = x^2\sin x - 2\,(x\,(-\cos x) - (-\sin x)) + C = x^2\sin x + 2x\cos x - 2\sin x + C$

I'll omit the $+C$ for the definite integral. When $x = \frac{\pi}{2}$, this expression is $\frac{\pi^2}{4} - 2$. When $x = 0$, this expression is 0. I've of course used the values of the trig functions at $\frac{\pi}{2}$ and 0. This difference of numbers is the answer we expected.

(16) 3. Suppose $\mathcal{R}$ is the region in the first quadrant bounded by the curves $y = \frac{1}{x}$, $y = \frac{x^2}{27}$ and $x = 1$.

Answer



I would begin this problem by drawing a picture with all the curves. We clearly need to know where $y = \frac{1}{x}$ and $y = \frac{x^2}{27}$ intersect, so we should solve $\frac{1}{x} = \frac{x^2}{27}$. This equation becomes $x^3 = 27$ with the pleasant solution $x = 3$.

a) Compute the area of $\mathcal{R}$.

Give an **exact** *answer, which may be an expression like "$\sqrt{7}\ln 5 + \pi$." A* **numerical** *approximation like "4.25817" is* **not acceptable**. *Techniques of calculus must be used.*

Answer The area is $\int_{\text{Left}}^{\text{Right}} \text{Top} - \text{Bottom}\, dx = \int_1^3 \frac{1}{x} - \frac{x^2}{27}\, dx = \ln x - \frac{x^3}{81}\Big]_1^3 = \ln 3 - \frac{27}{81} - \left(\ln 1 - \frac{1}{81}\right) = \ln 3 - \frac{26}{81}$.

b) The region $\mathcal{R}$ is revolved about the $x-$axis. Set up a definite integral which gives the volume of the resulting solid. Describe the method you are using.

DO NOT EVALUATE THE INTEGRAL.

Answer The washer method (a thin disc minus another thin disc) gives $\pi \int_1^3 \left(\frac{1}{x}\right)^2 - \left(\frac{x^2}{27}\right)^2 dx$.

c) The region $\mathcal{R}$ is revolved about the $y-$axis. Set up a definite integral which gives the volume of the resulting solid. Describe the method you are using.

DO NOT EVALUATE THE INTEGRAL.

Answer The thin shell method gives $\int_1^3 2\pi x \left(\frac{1}{x} - \frac{x^2}{27}\right) dx$.

(12) 4. Find the solution of the differential equation

$$x^2 \frac{dy}{dx} + y = 0$$

satisfying the initial condition $y(1) = 2$.

Answer This is a separable differential equation. Rewrite it as $\frac{dy}{dx} = \frac{-y}{x^2}$ and then as $\frac{dy}{y} = -\frac{dx}{x^2}$. Integrate to get $\ln y = \frac{1}{x} + C$. Of course the value of C is found using the given initial conditions: $\ln 2 = \frac{1}{1} + C$ so $C = \ln 2 - 1$. The solution we want is given by $\ln y = \frac{1}{x} + \ln 2 - 1$. For a more explicit formula we can exponentiate both sides and do some simple algebra: $y = e^{1/x + \ln 2 - 1} = e^{1/x} e^{\ln 2} e^{-1} = \frac{2}{e} e^{1/x}$.

If y is this specific solution, what is $\lim_{x \to +\infty} y(x)$?

Answer $\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} \frac{2}{e} e^{1/x} = \frac{2}{e}$. This is because the exponential function's values at small positive numbers are close to 1.

If y is this specific solution, what is $\lim_{x\to 0+} y(x)$?

Answer $\lim_{x\to 0+} y(x) = \lim_{x\to 0+} \frac{2}{e} e^{1/x} = +\infty$. This is because the exponential function's values at large positive numbers is very large positive.

(8) 5. Evaluate the improper integral, if it converges.

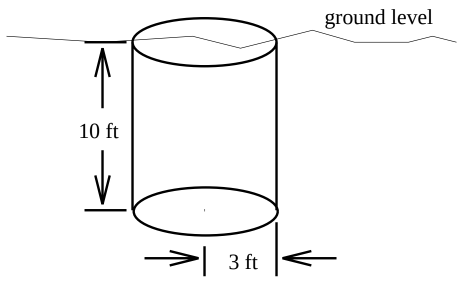
$$\int_0^4 \frac{dx}{x^3}$$

Answer This is improper because $\lim_{x\to 0+} \frac{1}{x^3} = +\infty$. So we handle it this way: $\int_0^4 \frac{dx}{x^3} = \lim_{A\to 0+} \int_A^4 \frac{dx}{x^3} =$
$= \lim_{A\to 0+} \int_A^4 x^{-3}\, dx = \lim_{A\to 0+} -\frac{1}{2} x^{-2}\Big]_A^4 = \lim_{A\to 0+} -\frac{1}{2}4^{-2} + \frac{1}{2}A^{-2} = +\infty$ and therefore the integral **diverges**.

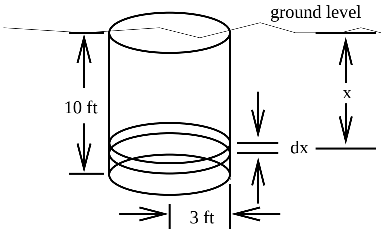
(10) 6. Does $\int_2^\infty \frac{4\,dx}{e^{2x}+x^2}$ converge? If it does, is its value less than $\frac{1}{4}$?

Answer I don't think an explicit antiderivative for $\frac{4}{e^{2x}+x^2}$ in terms of "known" functions can be found. But $\frac{4}{x^2}$ and $\frac{4}{e^{2x}}$ do have nice antiderivatives. Both of them can serve to compare with the integral given: they are both *larger* than the integral given. But this is a little tricky, because $\int_2^\infty \frac{4}{x^2}\, dx = \lim_{A\to\infty} \int_2^A \frac{4}{x^2}\, dx =$ $\lim_{A\to\infty} -\frac{4}{x}\Big]_2^A = \lim_{A\to\infty} -\frac{4}{A} + \frac{4}{2} = 2$. Thus the integral does indeed **converge** (since all of the functions involved are positive, and a larger improper integral than the one we were given converges). But this comparison tells us that the value of the integral must be less than 2, and doesn't give enough information to decide if it is less than $\frac{1}{4}$. Now let's compare it to the other candidate. That is, the integral we want to analyze is also less than $\int_2^\infty \frac{4}{e^{2x}}\, dx = \lim_{A\to\infty} \int_2^A \frac{4}{e^{2x}}\, dx = \lim_{A\to\infty} \int_2^A 4e^{-2x}\, dx = \lim_{A\to\infty} -2e^{-2x}\Big]_2^A = \lim_{A\to\infty} -2e^{-2A} + 2e^{-4} = \frac{2}{e^4}$. Now *this* number is about .037 which is surely less than $\frac{1}{4}$. So the integral does converge, and its value is less than $\frac{1}{4}$. What luck! Not really because it is a question on a calculus test.

(12) 7. A cylindrical tank of toxic waste has its axis of symmetry perpendicular to the ground. The top of the tank is at ground level, and its bottom is 10 ft below the ground. The radius of the cylindrical tank is **3 ft**. The toxic waste inside varies in its density so that at depth x ft from ground level ($0 \le x \le 10$) the density is $(20+4x)$ lb/ft^3. Show that 21 000 π ft-lbs of work is needed to pump the contents of the tank to ground level.



Answer Let's look at a thin slice of the toxic waste at a depth of x ft from ground level. The slice is dx thick, and its cross-sectional area is $\pi(3)^2$. Therefore the volume of the slice is $9\pi dx$ ft^3. The weight of the slice is density·volume, which is $(20+4x)9\pi\ dx$ lb. The work needed to raise this a distance of x ft to ground level is $x(20+4x)9\pi\ dx$ ft-lbs because work = force·distance. We must *add* all of these little pieces of work together (integrate!) to find the total work required to empty all of the tank.



The work required is $\int_0^{10} x(20+4x)9\pi\ dx = 9\pi\int_0^{10} 20x+4x^2\ dx = 9\pi\left(10x^2+\frac{4}{3}x^3\right)\Big]_0^{10} =$

$9\pi\left(10\cdot(10)^2+\frac{4}{3}(10)^3\right) = 9\pi(1\ 000)\left(1+\frac{4}{3}\right) = 9\ 000\pi\left(\frac{7}{3}\right) = 21\ 000\ \pi$ ft-lbs.

(12) 8. Compute the following integrals by finding antiderivatives and using the Fundamental Theorem of Calculus.

Give an **exact** *answer, which may be an expression like "$\sqrt{7}\ln 5+\pi$." A* **numerical** *approximation like "4.25817" is* **not acceptable**. *Techniques of calculus must be used.*

a) $\int_0^{\pi/4} (\sec x)^4\ dx$

Answer $\int_0^{\pi/4} (\sec x)^4\,dx = \int_0^{\pi/4} (\sec x)^2(\sec x)^2\,dx = \int_0^{\pi/4}\left(1+(\tan x)^2\right)d(\tan x) = \tan x+\frac{1}{3}(\tan x)^3\Big]_0^{\pi/4}$

and this is, of course, $\frac{4}{3}$ since $\tan 0 = 0$ and $\tan(\pi/4) = 1$.

Comment Of course I am abbreviating the substitution process in my answers to both parts. I should "officially" write $u=\tan x$ in a) followed by $du = (\sec x)^2\ dx$ and then in b) write $u = 7x^4$, etc.

b) $\int_0^1 x^3 e^{7x^4}\ dx$

Answer $\int_0^1 x^3e^{7x^4}\ dx = \int e^{7x^4}\frac{1}{28}\,d\left(7x^4\right) = \frac{1}{28}e^{7x^4}\Big]_0^1 = \frac{1}{28}\left(e^7-1\right)$.

(8) 9. The Trapezoidal Rule will be used to estimate the definite integral

$$\int_0^1 \sin\left(1+x^4\right)\ dx.$$

Find n (the number of subdivisions) so that the Trapezoidal Rule estimate will be within 10^{-6} of the true value of the definite integral. (You may use the error bound $\frac{K(b-a)^3}{12n^2}$ where K is an overestimate of the magnitude of the second derivative.)

DO NOT COMPUTE THE TRAPEZOIDAL RULE ESTIMATE.

Answer The first derivative of $\sin\left(1+x^4\right)$ is $\cos\left(1+x^4\right)4x^3$, so the second derivative of the function is $-\sin\left(1+x^4\right)16x^6+\cos\left(1+x^4\right)12x^2$. Now let us assume the *worst*: that the signs of the cos and sin don't allow any cancellation. But their magnitudes are no more than 1, and the largest that $16x^6$ and $12x^2$ can be on the interval $[0,1]$ are 16 and 12 respectively. Thus the magnitude of the second derivative is certainly no more than 28, and we can take this as K. We are therefore looking for an n so that $\frac{28}{12n^2}\le 10^{-6}$. Therefore we want n so that $28\ 000\ 000 \le 12n^2$ and this will be true (a reliable calculator tells me) if $1528 \le n$. In fact, this analysis tells me that *any* n larger than 1528 is a correct answer to this problem.

Note `Maple`'s sophisticated numerical integration program reports .8980831616 as the "true" value of this integral, and gives .8980831023 as the Trapezoid Rule approximation when $n = 1528$. Indeed.